- Statistical modeling: Regard y_{ij} as a realization of a r.v. Y_{ij}

For $i = 1, \ldots, I$; $j = 1, \ldots, J$,

parallel lines in interaction plot

$$Y_{ij} = \bar{\mu} + \alpha_i + \beta_j + \epsilon_{ij}, \quad (\leftarrow \text{ MEM}) \qquad (\Diamond)$$

$\cancel{\delta_{ij}}$ ⇒ model (□) in LNp.45 ← 2 treatment factors

where
- $\bar{\mu}$: grant mean
- α_i: effect of the ith treatment ($\longleftarrow$ main interest in RBD)
- β_j: effect of the jth block ($\longleftarrow$ no interest in RBD)
- ϵ_{ij}'s (random errors) $\sim$ i.i.d. $N(0, \sigma^2)$

規律 I II

隨機

$\Sigma_i \alpha_i = 0$

$\Sigma_j \beta_j = 0$

- **Q**: Why no interactions (i.e., δ_{ij}'s) in this model? dimension of them $= IJ$

# of all observations $= IJ$

There is not enough degrees of freedom to study interactions

(**Note.** If the model contains interactions, after $\bar{\mu}$, α_i's, β_j's, δ_{ij}'s are estimated, there is no degrees of freedom left for $\hat{\epsilon}_{ij}$'s.

$\hat{\mu}_{ij}$ under model (□)

$\Rightarrow$ all the residuals $\hat{\epsilon}_{ij} = Y_{ij} - \bar{Y}_{ij} = 0 \quad \Rightarrow \hat{\sigma}^2 = s_p^2 = 0$)

- In practice, it is commonly seen that there is no interaction between treatment and block factors due to the nature of block factors.

Example 8 (Examples of randomized block designs)

→ stratifying exp'tal units

a land

a plot

- To compare effects of I different fertilizers, relatively homogeneous blocks of land are selected and each is divided into I plots. Within each block, assign the fertilizers to plots at random. a land → a block, a plot → an exp'tal unit

a litter → a block, an animal → an exp'tal unit

- To compare effects of different I diets on experimental animals, select I animals from each of several litters and randomly assign to the diets.
- If an experiment is to be carried out over a substantial period of time, the blocks may be stretches of time. block exp'tal unit time
- In industrial experiments, the blocks may be batches of raw materials.

a batch → a block

one element of a batch → an exp'tal unit

Theorem 19 (ANOVA for randomized block design)

Consider the model ($\Diamond$) in LNp.65. Under this model, Thm 15 (LNp.50), Thm 16 (LNp.51), and Thm 17 (LNp.53) still hold, because this model is a sub-model (Ω^*) of the model (□) (Ω) in LNp.45 for 2-way layout. Thus, (LNp.64)

adding constraints all δ_{ij}'s $= 0$

Ω^*

In 1-way layout, $\hat{\sigma}^2 = (SS_B + SS_{AB})/(df_B + df_{AB})$ ← cf.

$$\hat{\sigma}^2 = MS_{AB} = SS_{AB}/[(I-1)(J-1)]$$

$Y_{ij} = \bar{\mu} + \alpha_i + \beta_j + \varepsilon_{ij} = \hat{\bar{\mu}} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\delta}_{ij}$

- (from Thm 15 in LNp.50) $SS_{TOT} = SS_A + SS_B + SS_{AB}$, where SS_{AB} plays the role of SS_E since all δ_{ij}'s $= 0$.

$+ \cancel{SS_E}$ ($\because$ one obs. in a cell, $K=1$)

What role?

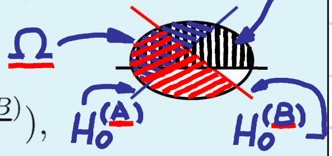
SS_E in Thm 16 (LNp.52)

- (from Thm 16 in LNp.51 by setting $K = 1$ and all δ_{ij}'s $= 0$)
  - $E(SS_A) = J(\sum_i \alpha_i^2) + (I-1)\,\sigma^2 \quad \Leftrightarrow \quad E(MS_A) = \frac{J}{I-1}(\sum_i \alpha_i^2) + \sigma^2$ ← ≥ 0
  - $E(SS_B) = I(\sum_j \beta_j^2) + (J-1)\,\sigma^2 \quad \Leftrightarrow \quad E(MS_B) = \frac{I}{J-1}(\sum_j \beta_j^2) + \sigma^2$ ← ≥ 0

cf

$E(SS_{AB}) = (I-1)(J-1)\,\sigma^2 \quad \Leftrightarrow \quad E(MS_{AB}) = \sigma^2$

- (from Thm 17 in LNp.53-54) → Note. $\Omega^* \subset \Omega$
  - Under $H_0^{(A)}: \alpha_1 = \cdots = \alpha_I = 0$ (Note. $H_A^{(A)}: \Omega^* \backslash H_0^{(A)}$), ($\delta_{ij}$'s$=0$) Ω^*
  $$SS_A \sim \sigma^2 \chi^2_{I-1} \quad \Leftrightarrow \quad SS_A/\sigma^2 \sim \chi^2_{I-1}.$$
  - Under $H_0^{(B)}: \beta_1 = \cdots = \beta_J = 0$ (Note. $H_A^{(B)}: \Omega^* \backslash H_0^{(B)}$),
  $$SS_B \sim \sigma^2 \chi^2_{J-1} \quad \Leftrightarrow \quad SS_B/\sigma^2 \sim \chi^2_{J-1}.$$



  - Under $\Omega^* = H_0^{(A)} \cup H_A^{(A)} = H_0^{(B)} \cup H_A^{(B)}$, ← all δ_{ij}'s $=0$
  $$SS_{AB} \sim \sigma^2 \chi^2_{(I-1)(J-1)} \quad \Leftrightarrow \quad SS_{AB}/\sigma^2 \sim \chi^2_{(I-1)(J-1)}.$$
  - Under Ω^*, the SS_A, SS_B and SS_{AB} are independent.

for treatment factor

- Test $H_0^{(A)}: \alpha_1 = \cdots = \alpha_I = 0$ vs. $H_A^{(A)}$: at least one of α_i's is not 0
  - test statistic $F_A = \dfrac{MS_A}{MS_{AB}} = \dfrac{SS_A/(I-1)}{SS_{AB}/[(I-1)(J-1)]}$. ← justification from $E(MS_A)$ & $E(MS_{AB})$ in LNp.66 ⇒ larger F_A, more extreme

    MS_E in LNp.56 $\xleftrightarrow{cf.}$ MS_{AB}
  - null distribution: under $H_0^{(A)}$, → $\sim \chi^2_{I-1}$ (under $H_0^{(A)}$)
  $$F_A = \frac{(SS_A/\sigma^2)/(I-1)}{(SS_{AB}/\sigma^2)/[(I-1)(J-1)]} \sim F_{I-1,(I-1)(J-1)}$$
  independent

  $\chi^2_{(I-1)(J-1)}$ (under Ω^*) $\sim$
  - rejection region at level α: reject $H_0^{(A)}$ if $F_A > F_{I-1,(I-1)(J-1)}(\alpha)$

for block factor

- Test $H_0^{(B)}: \beta_1 = \cdots = \beta_J = 0$ vs. $H_A^{(B)}$: at least one of β_j's is not 0
  - test statistic $F_B = \dfrac{MS_B}{MS_{AB}} = \dfrac{SS_B/(J-1)}{SS_{AB}/[(I-1)(J-1)]}$ ← justification same as that for F_A. ⇒ larger F_B, more extreme

    MS_E in LNp.56 $\xleftrightarrow{cf.}$ MS_{AB}
  - null distribution: under $H_0^{(B)}$, → $\sim \chi^2_{J-1}$ (under $H_0^{(B)}$)
  $$F_B = \frac{(SS_B/\sigma^2)/(J-1)}{(SS_{AB}/\sigma^2)/[(I-1)(J-1)]} \sim F_{J-1,(I-1)(J-1)}$$
  independent

  $\chi^2_{(I-1)(J-1)}$ (under Ω^*) $\sim$
  - rejection region at level α: reject $H_0^{(B)}$ if $F_B > F_{J-1,(I-1)(J-1)}(\alpha)$
- ANOVA table for RBD ($K=1$ $\xleftrightarrow{cf.}$ ANOVA table for 2-way layout (LNp.57))

規律

Source	SS	df	MS (=SS/df)	F
I (treatment) A	SS_A	$I-1$ $\sum_i \alpha_i = 0$	$SS_A/(I-1)$	MS_A/MS_{AB}
II (block) B	SS_B	$J-1$ $\sum_j \beta_j = 0$	$SS_B/(J-1)$	MS_B/MS_{AB}
(error) AB	SS_{AB}	$(I-1)(J-1)$	$SS_{AB}/[(I-1)(J-1)]$ $=\hat{\sigma}^2 \xrightarrow{e} \sigma^2$	use same standard
Total	SS_{TOT}	$IJ-1$ ← $\hat{\mu}$		

隨機

Example 9 (Drugs to relieve itching, Beecher, 1959)

- Five drugs were compared to a placebo and no drug with ⇒ treatment factor A, $I=7$ ← of main interest (通則) of no interest (特例)
- 10 volunteer male subjects aged 20-30 ⇒ block factor B, $J=10$ ←

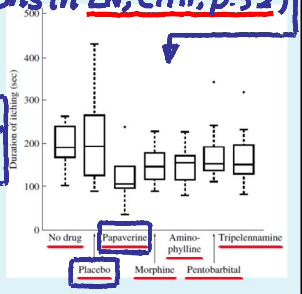
In 1-way layout, the 2 lines merge to form "within" (check table in LNp.18) ⇒ $\hat{\sigma}^2$ becomes larger in this case

- Each volunteer underwent one treatment per day, and the time-order was randomized.
- The subjects were given a drug or placebo intravenously, and then itching was induced on their forearms with cowage. The subjects recorded the duration of the itching.
- Analysis of variance table for RBD

blocks — subject 1 … subject 10; homogeneous within, heterogeneous between

r treatments 1, …, 7 r; day 1 • • • day 7; an exp'tal unit

Note. Variation of subjects still appears in these boxplots (cf., graphs in LN, CH11, p.52)



Source	SS	df	MS	F
規 (treatment) Drugs (A)	53013	6	8835	2.85
規 (block) Subjects (B)	103280	9	11476	3.71 significant, but not interested
隨 (error) Interaction	167130	54	3095 ($\hat{\sigma}^2$)	
Total	323422	69 = 70-1		

multiple comparison

The F statistic for testing differences between drugs is 2.85, corresponding to a p-value less than 0.025 ⇒ significant differences.

- Tukey's simultaneous 95% confidence intervals for all differences between drugs have half-widths of — variation of subjects has been removed

Check Thm 11 (LNp.31~34) for 1-way layout → $q_{7,54}(0.05)(s/\sqrt{J}) = 4.31\sqrt{3095/10} = 75.8.$

Only papaverine achieves a reduction of itching over the effect of a placebo.

Note 9 (Some notes about ANOVA for RBD) — at least one of δ_{ij}'s $\neq 0$

- From Thm 16 in LNp.51, if there is an interaction between A and B,

harder to reject $H_0^{(A)}$ ∵ $E(MS_{AB}) > \sigma^2 \Rightarrow F_A \downarrow$

$$E(MS_{AB}) = \sigma^2 + \frac{1}{(I-1)(J-1)}\left(\sum_i\sum_j \delta_{ij}^2\right) > \sigma^2$$

and the actual probability of type I error will be smaller than desired.

cf. 2-sample T is a special case of 1-way ANOVA (LNp.22)

- Connection between the ANOVA for RBD and the paired t test (Thm 18, LN, CH11, p.53-54): the pair t test is a special case of the ANOVA for RBD where only two treatments are being compared ($I=2$). Here are some hints for a proof (exercise): check the graphs in LNp.63
  - **recall**: In pair t test, the data is n pairs of (X_j, Y_j) and $D_j = X_j - Y_j$
  - Let $Z_{1j} = X_j$, $Z_{2j} = Y_j$, and consider the model (◊) in LNp.65 for Z_{ij}'s.
  - We have $\overline{Z}_{1.} = \overline{X}$, $\overline{Z}_{2.} = \overline{Y}$, $\overline{Z}_{..} = (\overline{X}+\overline{Y})/2$, and $\overline{Z}_{.j} = (X_j+Y_j)/2$
  - $MS_A = \sum_{i=1}^2(\overline{Z}_{i.}-\overline{Z}_{..})^2 = (\overline{X}-\overline{Y})^2/2$ and $MS_{AB} = \frac{1}{n-1}\sum_{i=1}^2\sum_{j=1}^n(Z_{ij} - \overline{Z}_{i.} - \overline{Z}_{.j} + \overline{Z}_{..})^2 = \frac{1}{2(n-1)}\sum_{j=1}^n[\underbrace{(X_j-Y_j)}_{D_j} - \underbrace{(\overline{X}-\overline{Y})}_{\overline{D}}]^2$

reject if $|T|>c$ ⇔ $F_A > c^2$

  - Then,

$$F_A = \frac{MS_A}{MS_{AB}} = \left(\frac{\overline{X}-\overline{Y}}{\frac{1}{n-1}\sum_{j=1}^n[(X_j-Y_j)-(\overline{X}-\overline{Y})]^2}\right)^2 = \left(\frac{\overline{D}}{s_{\overline{D}}}\right)^2 = T^2$$

• A nonparametric test --- the Friedman's test --- for RBD

Kruskal-Wallis test (no blocks) in LNp. 23~25

- Consider the model (◊) in LNp.65. ← no interaction (δ_{ij}'s $= 0$)
- But, assume that all the errors ϵ_{ij}'s $\sim$ i.i.d. $F(x)$, ← $\dim(\{F\}) = \infty$ where $F(x)$ can be any distribution with mean zero (or median zero).
- Then, $Y_{ij} \sim F(x - \mu_{ij}) = F(x - (\bar{\mu} + \alpha_i + \beta_j))$ and all Y_{ij}'s are independent.

cf.

Theorem 20 (Friedman's test) within-block ranking ⇒ can remove block effects

- (Replace Y_{ij}'s by R_{ij}'s) Within each of the J blocks, the observations are ranked. Within the jth block, $j = 1, \ldots, J$, let R_{ij} be the rank of the observation pertaining to the ith treatment, $i = 1, \ldots, I$, i.e.,

  j is fixed → $R_{1j}, \cdots, R_{Ij}$ are the ranks of $Y_{1j}, \cdots, Y_{Ij}$

- An example: treatment i (α_i) Y_{ij}'s → R_{ij}'s

(A) (B) block j (β_j)

Subject	No Drug	Placebo	Papa-verine	Morphine	Amino-phylline	Pento-barbital	Tripelen-namine
BG	174	263	105	199	141	108	141
JF	224	213	103	143	168	341	184
BS	260	231	145	113	78	159	125
SI	255	291	103	225	164	135	227
BW	165	168	144	176	127	239	194
TS	237	121	94	144	114	136	155
GM	191	137	35	87	96	140	121
SS	100	102	133	120	222	134	129
MU	115	89	83	100	165	185	79
OS	189	433	237	173	168	188	317
Average	191.0	204.8	118.2	148.0	144.3	176.5	167.2

cf.

← $\bar{Y}_{i\cdot}$ (⇒ SS_A)

	No Drug	Placebo	Papaverine	Morphine	Amino-phylline	Pento-barbital	Tripelen-namine
BG	5	7	1	6	3.5	2	3.5
JF	6	5	1	2	3	7	4
BS	7	6	4	2	1	5	3
SI	6	7	1	4	3	2	5
BW	3	4	2	5	1	7	6
TS	7	3	1	5	2	4	6
GM	7	5	1	2	3	6	4
SS	1	2	5	3	7	6	4
MU	5	3	2	4	6	7	1
OS	4	7	5	2	1	3	6
Average	5.10	4.90	2.30	3.50	3.05	4.90	4.25

cf.

← $\bar{R}_{i\cdot}$ (⇒ $SS_A^{(R)}$)

- Null hypothesis: there is no effect due to the treatment factor, i.e.,

$$H_0^{(A)} : \alpha_1 = \cdots = \alpha_I = 0$$

- Apply sum of squares decomposition on R_{ij}'s:

$$\underbrace{SS_{TOT}^{(R)}}_{\substack{= \text{ a constant} \\ \text{if no tie}}} = SS_A^{(R)} + \underbrace{SS_B^{(R)}}_{=0} + SS_{AB}^{(R)} \quad \Rightarrow SS_{AB}^{(R)} \text{ is a function of } SS_A^{(R)}$$

- Test statistic: $SS_A^{(R)} = J\sum_{i=1}^{I}(\bar{R}_{i\cdot} - \bar{R}_{\cdot\cdot})^2$. (Q: Why not divided by $SS_{AB}^{(R)}$?)

  larger $SS_A^{(R)}$ ⇒ more extreme

- Null distribution of $SS_A^{(R)}$
  - Under $H_0^{(A)}$, the exact distribution of $SS_A^{(R)}$ can be calculated as follows.
    * When $\alpha_1 = \cdots = \alpha_I = 0$, for the data in block j, $j = 1, \ldots, J$, we have (j is fixed)

      like 1-sample data → $Y_{1j}, \cdots, Y_{Ij} \sim$ i.i.d. $F(x - (\bar{\mu} + \beta_j))$ ← different dist. for different j's (blocks)

      ⇒ For any permutations r_{ij}'s of $\{1, \ldots, I\}$, cf. same dist. for different j's (blocks)

      ranks $1, \ldots, I$ without replacement $R_{1j} \cdots R_{Ij}$

      $$P(R_{1j} = r_{1j}, \ldots, R_{Ij} = r_{Ij}) = 1/(I!)$$ ← → irrelevant to F & β_j's

    * The J collections of ranks $\{R_{11}, \ldots, R_{I1}\}$, $\ldots$, $\{R_{1J}, \ldots, R_{IJ}\}$ are independent. → joint dist. $= \prod_{j=1}^{J}$ jth marginal dist.
    * From the null distribution of R_{ij}'s, the exact null distribution of $SS_A^{(R)}$ can be enumerated.

– Or, we can use this asymptotic null distribution $SS_A^{(R)} \overset{D}{\approx} \frac{I(I+1)}{12}\chi^2_{I-1}$

Why? Check Kruskal-Wallis test stat. in LN p.24~25

$$Q = \frac{\boxed{12} SS_A^{(R)}}{\boxed{I(I+1)}} = \frac{12\,J}{I(I+1)} \sum_{i=1}^{I} (\bar{R}_{i.} - \bar{R}_{..})^2 \overset{D}{\approx} \chi^2_{I-1}.$$

Note 10 (Some notes about the Friedman's test)

- Connection between Friedman's test and the sign test for paired samples (LN, CH11, p.57-58): the sign test is a special case of Friedman's test where only 2 treatments are being compared ($I = 2$) (exercise).
- For your information, there is a nonparametric test that generalizes the signed rank test (Thm 19, LN, CH11, p.58-61) from 2 treatments to I treatments. It is called Quade test.

	$j=1$	$j=2$	$\cdots$	$j=J$	
$i=1$	$Y_{11} \to R_{11}=2$	$Y_{12} \to R_{12}=1$	$\cdots$	$Y_{1J} \to R_{1J}=1$	$\to \bar{R}_{1.} = [1\times(J-N_+)+2\times N_+]/J$
	$\vee$	$\wedge$		$\wedge$	
$i=2$	$Y_{21} \to R_{21}=1$	$Y_{22} \to R_{22}=2$	$\cdots$	$Y_{2J} \to R_{2J}=2$	$\to \bar{R}_{2.} = [1\times N_+ + 2(J-N_+)]/J$
	$D_1>0$	$D_2<0$	$\cdots$	$D_J<0$	$\to N_+ = \#\{D_j>0\}$

$SS_A^{(R)} > c \Leftrightarrow \left|N_+ - \frac{J}{2}\right| > c^*$ ← sign test

❖ **Reading**: textbook, 12.3.3, 12.3.4

❖ **Further reading**: textbook, 12.4 (**factorial design**, **fractional factorial designs**, **incomplete block designs**, **assumptions in ANOVA**)