- A nonparametric method --- the signed rank test

allows dist. with variance $=\infty$, such as Cauchy.

- Let Ω be the collection of *all* continuous distributions $\Rightarrow \dim(\Omega)=\infty$
- Consider the nonparametric statistical model: $D_1, \ldots D_n \sim$ i.i.d. F, $\quad (\nabla)$
  where $F \in \Omega$.

$P(D>d)=P(D<-d),\ \forall d \in \mathbb{R}$

cf. $\mu_D = 0$ ($\mu_X - \mu_Y$)

- Let $\Omega_0 = \{F \mid F \in \Omega$ and F is symmetric about $0\}$ $\quad$ $\boxed{P(D>0)=P(D<0)=1/2}$
  - $\Omega_0 \subset \Omega$ and $\dim(\Omega_0)=\infty$ $\leftarrow$ odd-order moments $=0$, even-order moment $\in \mathbb{R}$
  - If $F \in \Omega_0$, then the median of F is 0. But, F with median zero is not necessary a distribution being symmetric about 0.
- Under the model (∇), we want to test the null and alternative hypotheses:

$$H_0: F \in \Omega_0 \quad \text{vs.} \quad H_A: F \in \Omega \setminus \Omega_0$$

symmetric about 0 $\Rightarrow$ median $=0$; $\not\Leftarrow$

Q: Why add the "symmetric" condition in the null?

always exists

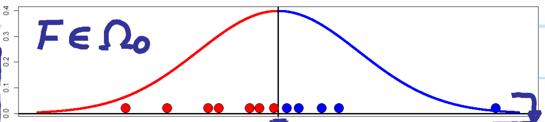
Question 8.

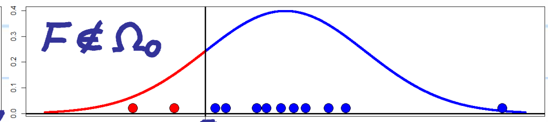
How to use ranks to examine "symmetric about 0"? What data are "more extreme," i.e., cast more doubts on H_0?

cf. **Intuition.**

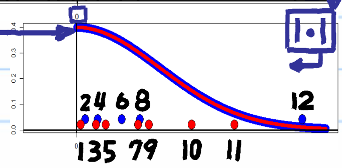
rank sum W_X or W_Y test (LNp.32): $H_0: F=G$

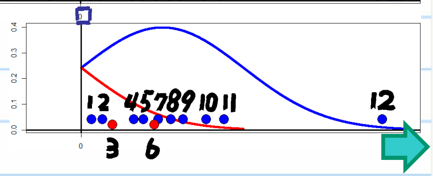
This explains why add symmetric condition





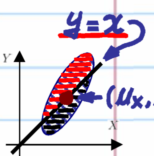
ranks:





- A brief comparison to (2-sample) rank sum test (assume no $D_i = 0$) $\quad$ $\because F$ is continuous
  - similarity: if in the paired case, $\quad$ W_X, W_Y (LNp.32)
    * the data $\{-D_i \mid D_i < 0\}$ is treated as the 1st sample
    * the data $\{D_i \mid D_i > 0\}$ is treated as the 2nd sample
    * then, the calculation for the paired case is equivalent to the rank-sum statistic in the unpaired case

$P(D>0) \overset{?}{=} 1/2$
$= P(X-Y>0)$
$= P(X>Y)$
$\overset{?}{=} P(X<Y)$
$= P(X-Y<0)$
$= P(D<0) \overset{?}{=} 1/2$



$\mu_X = \mu_Y$
$\Delta = 0$
(Recall. π in LNp.36)

  - difference
    * In 2-sample unpaired cases, the sample sizes m, n are fixed numbers.
    * In the paired case, the sizes

What information contained in N_+ & N_-? cf.

$$N_- = \#\{D_i < 0\} \quad \text{and} \quad N_+ = \#\{D_i > 0\}$$

(**Note.** $N_- + N_+ = n$) are random variables.

Under alternative, $I_{[D_1>0]}, \cdots, I_{[D_n>0]}$ i.i.d. Bernoulli$(1-F(0))$ ← cdf of D

Indicator function: $I_{[D_i>0]} = \begin{cases} 1, & D_i>0 \\ 0, & D_i \le 0 \end{cases}$

    * Under H_0, ← F is symmetric about 0
      - $I_{[D_1>0]}, \ldots, I_{[D_n>0]} \sim$ i.i.d. Bernoulli$(1/2)$, cf.
      - $N_+ = \sum_{i=1}^n I_{[D_i>0]} \sim \text{bin}(n, 1/2)$ and $N_- = n - N_+ \sim \text{bin}(n, 1/2)$
    * When conditioned on N_+ (or N_-), the null distribution of the test statistic in the paired case is identical to the null distribution of rank-sum statistic in the unpaired case. ← conditional

N_+ & N_- are known

- Alternative test: sign test (TBp.461, problem 12) $\overset{cf.}{\longleftrightarrow}$ Mann-Whitney test U_X, U_Y (LNp.38)
  - Consider the model (∇) in LNp.56.

– Let $\Omega_0^* = \{F \mid F \in \Omega$ and F has median $0\}$

$*$ $\Omega_0 \subset \Omega_0^* \subset \Omega$ and $\dim(\Omega_0^*) = \infty$ ← symmetric about 0 $\cancel{\Leftarrow}\Rightarrow$ median = 0

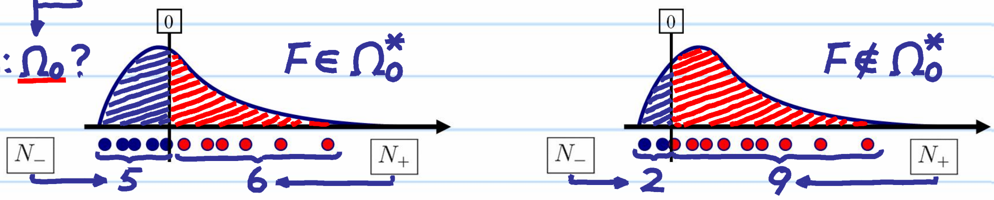
– Under the model (∇), test the null and alternative hypotheses:

cf. $H_0^* : F \in \Omega_0^*$ vs. $H_A^* : F \in \Omega \setminus \Omega_0^*$

⊖ **Intuition.**

Is this good enough for $H_0 : \Omega_0$?

Note. W_+ & W_- use more information of data than N_+ & N_-



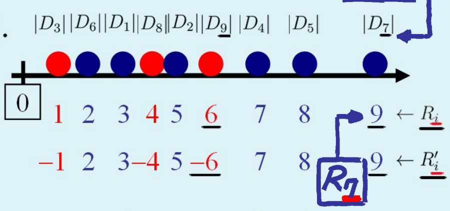
– Reject H_0^* if N_+ (or N_-) is small (close to 0) or large (close to n)

– Null distribution of N_+ (or N_-): bin(n, 1/2)

Theorem 19 (Wilcoxon signed rank test) → $W_- = \sum_i I_{[D_i<0]}(-R_i')$ R_i $i=7$

Consider the nonparametric model (∇) in LNp.56.

- test ststistic W_+ (or $W_- = \frac{n(n+1)}{2} - W_+$)



(1) Let R_i = rank of $|D_i|$, $i = 1, \ldots, n$.

(2) Restore the signs of D_i's to the ranks R_i''s, i.e., let $R_i' = \text{sign}(D_i) \times R_i$.

(3) $W_+ = \sum_{i=1}^n I_{[D_i>0]} R_i'$, i.e., sum of the ranks R_i''s that have positive signs.

R_i

- **Q**: What values of W_+ are more extreme? If there is no difference between the two paired conditions, we expect ↳ more support for H_A

– about *half* the D_i's to be positive and *half* negative (median=0?) ← Ω_0^*

$\cup$

check graph in LNp.56 → ⊖ positive R_i''s and negative R_i''s *similarly* distributed (symmetric?) ← Ω_0

and W_+ will not be too small or too lagre ↳ similar histograms R_i $I_{[D_i>0]}$

⇒ data with larger or smaller W_+ are more extreme ⇒ tend to reject H_0

- Null distribution of W_+

$P(D \geq d) = P(D \leq -d) = P(-D \geq d),\ \forall d$

$P(|D| \leq x) = 2P(0 < D \leq x) = 2[P(D \leq x) - P(D \leq 0)] = 2[F(x) - 1/2]$

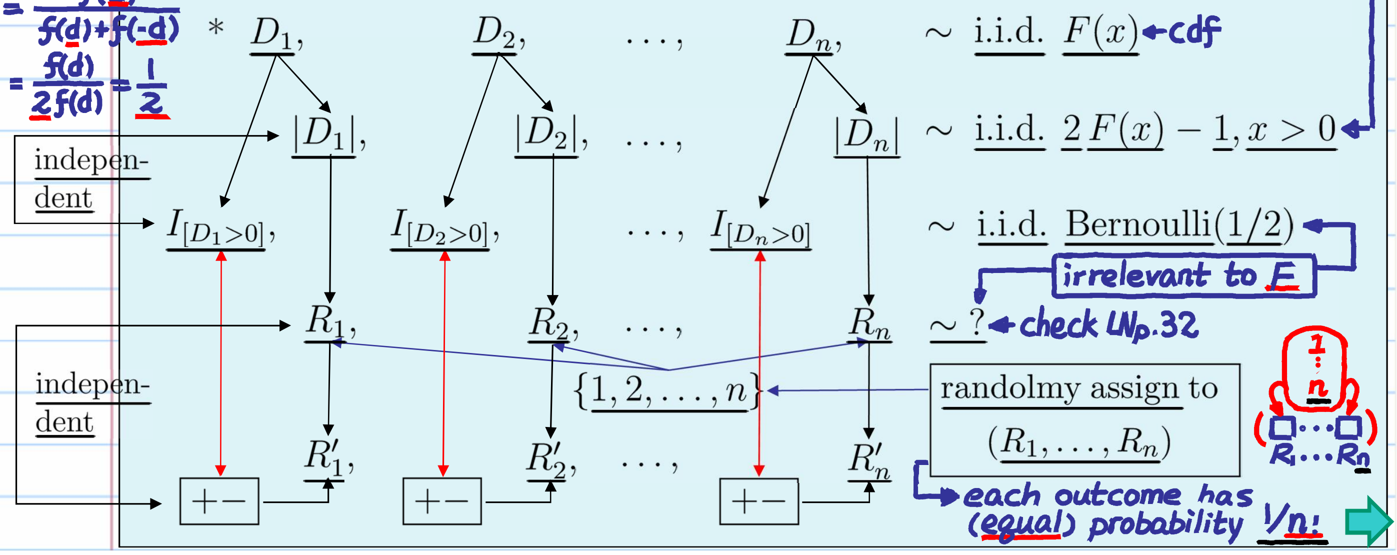
contain same information

– $W_+ \in \{0, 1, 2, \ldots, \frac{n(n+1)}{2}\}$

– Under the null H_0 (F is symmetric about 0)

$P(I=1 \mid |D|=d) = \frac{f(d)}{f(d)+f(-d)} = \frac{f(d)}{2f(d)} = \frac{1}{2}$

$*$ $D_i \Leftrightarrow (I_{[D_i>0]}, |D_i|)$ and D_i has the same distribution as $-D_i$



* Any particular assignment of $\{-, +\}$ signs to the integers $1, \ldots, n$ (the ranks) is equally likely.

Prob. (R_1', R_2', R_3')
$1/(2^3 \times 3!) \rightarrow (1, -2, 3)$
$1/(2^3 \times 3!) \rightarrow (1, 3, -2)$
$1/(2^3 \times 3!) \rightarrow (-2, 1, 3)$
$1/(2^3 \times 3!) \rightarrow (-2, 3, 1)$
$1/(2^3 \times 3!) \rightarrow (3, 1, -2)$
$1/(2^3 \times 3!) \rightarrow (3, -2, 1)$
$\vdots$

$\rightarrow$ same W_+ $\{1, -2, 3\}$

$\text{Prob.} = \dfrac{3!}{2^3 \times 3!} = 1/2^3$

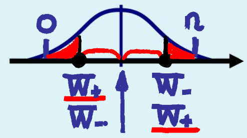
$\boxed{+\,-}$ $\boxed{R_i}$

$$\{R_i'\text{'s}\} = \{\ \overset{\boxed{1/2}}{\pm}\ 1,\ \overset{\boxed{1/2}}{\pm}\ 2,\ \ldots,\ \overset{\boxed{1/2}}{\pm}\ n\ \}$$

independent

$\text{Prob.} = \dfrac{n!}{2^n \times n!} = \dfrac{1}{2^n}$

* There are 2^n such assignments and for each we can calculate $W_+ \Rightarrow$ obtain 2^n values (not all distinct) of W_+, each with probability $1/2^n$.
* The probability of each distinct value of W_+ may thus be calculated, giving the desired null distribution.



check LNp.33

- (Two-sided) rejection region Recall. $W_+ + W_- = \dfrac{n(n+1)}{2}$
  (exercise) – The null distribution of W_+ is symmetric around $E(W_+) \rightarrow \dfrac{n(n+1)}{4}$

  cf. $\boxed{W \,\&\, W' \text{ in LNp.33}}$

  – Reject H_0 when $\min(W_+, W_-)$ is small, i.e., $\min(W_+, W_-) \le w$
  – Table 9 of Appendix B in textbook (TBp.A24) gives critical values w
- Ties
  – Tie between (X_i, Y_i): If some of the differences D_i's are zero, the most common technique is to discard those observations. $\boxed{D_i = 0}$

  – Tie between $|D_i|$'s: If there are ties, each $|D_i|$ is assigned the average value of the ranks for which it is tied.
  – If there are a large number of ties, modifications must be made. See Hollander and Wolfe (1973) or Lehmann (1975).

Example 9 (Smoking effect, signed-rank test for paired data, cont. Ex.7 In LNp.52)

- $n = 11$, $W_- = 1$ and $W_+ = [11(11+1)]/2 - W_- = 65 \Rightarrow \min(W_-, W_+) = 1$
- From Table 9 of Appendix B (TBp.A24), the critical value for two-sided test with significant level $\alpha = 0.01$ is 5.
- Since $\min(W_-, W_+) < 5$, reject H_0 at $\alpha = 0.01$ (consistent with the test result in Ex.8, LNp.55).

Note8 (LNp.39) cf.

Note 10 (A comparison of one-sample t-test and signed rank test for paired data)

- Unlike (one-sample) t-test, the signed-rank test does not depend on normality assumption. good for data with outliers ($\leftarrow$ Cauchy dist.)
- The signed-rank test is insensitive to outliers, whereas the t-test is sensitive.
- When the normality assumption holds, the t-test is more powerful. can check box-plot of D_i's
- However, it has been shown that even when normality assumption holds, the signed-rank test is nearly as powerful as the t-test (relatively efficiency of signed-rank test statistic to (one-sample) t-test statistic ≈ 0.95).
- The signed-rank test is generally preferable, especially for *small* sample sizes.

Theorem 20 (means and variances of W_+ under H_0)

- Consider the nonparametric model (∇) in LNp.56.
- Under the null hypothesis H_0: F is symmetric about 0,

$$\mathrm{E}(W_+) = \frac{n(n+1)}{4} \quad \text{and} \quad \mathrm{Var}(W_+) = \frac{n(n+1)(2n+1)}{24}.$$

same variance

$$\left(\Leftrightarrow \mathrm{E}(W_-) = [n(n+1)]/4 \quad \text{and} \quad \mathrm{Var}(W_-) = [n(n+1)(2n+1)]/24 \right.$$

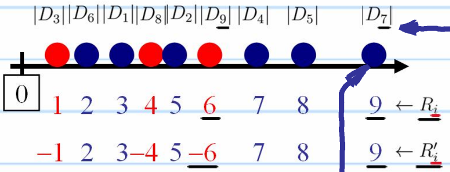
$$\left. \text{since } W_- = [n(n+1)]/2 - W_+ \right)$$

$R_1, \ldots, R_n$ indep (LNp.59)

Proof.

- For $k = 1, \ldots, n$, let $I_k = \begin{cases} 1, & \text{if the } k\text{th largest } |D_i| \text{ has } D_i > 0, \\ 0, & \text{otherwise.} \end{cases}$

k $\quad I_9 = I_{[D_7>0]}$ $\quad i$



- Under H_0, order statistic of $I_{[D_1>0]}, \ldots, I_{[D_n>0]}$ ← cf.

$I_{[D_1>0]}, \ldots, I_{[D_n>0]}$ → $I_1, \ldots, I_n \sim$ i.i.d. Bernoulli(1/2),
 – $E(I_k) = 1/2$ and $Var(I_k) = 1/4$.

- Write

$$W_+ = \sum_{i=1}^n I_{[D_i>0]} R_i' = \sum_{k=1}^n k\, I_k.$$

$i \to k(i)$: r.v. depending on R_i's

$k = 9$, $i = 7$

k : not random

I_k : r.v.

- Thus,

$$E(W_+) = \sum_{k=1}^n k\, E(I_k) = \frac{1}{2}\left(\sum_{k=1}^n k\right) = \frac{n(n+1)}{4}$$

$$Var(W_+) = \sum_{k=1}^n k^2\, Var(I_k) = \frac{1}{4}\left(\sum_{k=1}^n k^2\right) = \frac{n(n+1)(2n+1)}{24}$$

Theorem 21 (Asymptotic null distribution of W_+)

- Consider the nonparametric model (∇) in LNp.56.
- Under the null H_0 : F is symmetric about 0,
if the sample size n is greater than* 20, the null distribution of W_+ is well approximated by a normal distribution, i.e., ← cf. Thm13 (LNp.41)

$$\frac{W_+ - E(W_+)}{\sqrt{Var(W_+)}} \overset{D}{\approx} N(0,1) \quad \left(\text{or} \quad \frac{W_- - E(W_-)}{\sqrt{Var(W_-)}} \overset{D}{\approx} N(0,1) \right).$$

can be used to determine critical value & rejection region

Hint for Proof. Use the expression $W_+ = \sum_{k=1}^n k\, I_k$ to find the moment generating function of W_+, and show it converges (after standardization) to the moment generating function of $N(0,1)$, which is $e^{t^2/2}$. ← let $n \to \infty$

$$M_{W_+}(t) = E(e^{tW_+}) = E\left[e^{t(\sum_k k I_k)}\right]$$

$$= E\left[\prod_{k=1}^n e^{(tk) I_k}\right] = \prod_{k=1}^n E\left[e^{(tk) I_k}\right] = \prod_{k=1}^n \frac{1}{2}(1 + e^{tk})$$

$\because I_k \sim$ Bernoulli(1/2)

$\because I_1, \cdots, I_n$ indep.

❖ **Reading**: textbook, 11.3