- Test $H_0 : \pi_\Delta = 1/2$ ($\Leftrightarrow \Delta = 0$) vs. $H_A : \pi_\Delta \neq 1/2$ ($\Leftrightarrow \Delta \neq 0$)
  - intuitively, should reject H_0 if $\hat{\pi}_\Delta$ is too small (closer to 0) or too large (closer to 1)
  - test statistic U_Y (or U_X)

order statistics of $\{Y_1, \cdots, Y_m\}$ → $Y_{(1)}$ $Y_{(2)}$ $Y_{(3)}$ $Y_{(4)}$ $Y_{(5)}$ $Y_{(6)}$

pooled ranks. 1 2 3 4 5 6 7 8 9 10 11 12

$R_{Y_{(1)}}$ $R_{Y_{(2)}}$ $R_{Y_{(3)}}$ $R_{Y_{(4)}}$ $R_{Y_{(5)}}$ $R_{Y_{(6)}}$

j = 1 2 3 4 5 6 ranks in $\{Y_1, \cdots, Y_m\}$

  - ∗ Define

$$U_Y \equiv (mn)\,\hat{\pi}_\Delta = \sum_{i=1}^{n}\sum_{j=1}^{m} Z_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{m} V_{ij}\,.$$

W_Y ←cf.→ ⊙ Reject H_0 if U_Y is too small or too large (closer to 0 or mn).

  - ∗ Let $R_{Y_{(j)}}$ be the rank of $Y_{(j)}$ in the pooled sample. Then,

in $\{Y_1, \cdots, Y_m\}$ $\{X_1, \cdots, X_n, Y_1, \cdots, Y_m\}$

$$\sum_{j=1}^{m} R_{Y_{(j)}} = \text{rank sum of } Y_j\text{'s (or } Y_{(j)}\text{'s)} = R_{n+1} + \cdots + R_{n+m} = W_Y.$$

check LNp 32

  - ∗ Notice that

$$U_Y = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} V_{ij}\right) = \sum_{j=1}^{m} \underbrace{(R_{Y_{(j)}} - j)}_{\#\{X_{(i)} < Y_{(j)}\}} = \left(\sum_{j=1}^{m} R_{Y_{(j)}}\right) - \frac{m(m+1)}{2} = W_Y - [m(m+1)]/2.$$

$\#\{X_i \geq Y_j\}$ changed fixed

  - ∗ Similarly, U_X can be defined by changing "$X_{(i)} < Y_{(j)}$" in V_{ij} to "$X_{(i)} > Y_{(j)}$", and ← check graph in Thm10 (LNp.36)
    - · $\frac{1}{mn} U_X \xrightarrow{e} 1 - \pi_\Delta = P_\Delta(X > Y)$ ←cf.→ $\frac{1}{mn} U_Y \xrightarrow{e} \pi_\Delta = P_\Delta(X < Y)$

- · $U_X = mn - U_Y$ ←cf.→ $W_X = \frac{(m+n)(m+n+1)}{2} - W_Y$
- · $U_X = W_X - \frac{1}{2}n(n+1)$ ←cf.→ $U_Y = W_Y - \frac{1}{2}m(m+1)$
- · reject H_0 if U_X is too small or too large

  - null distribution of U_Y: the pmf of U_Y under H_0 can be obtained from the null distribution of W_Y by

pmf symmetric about $\frac{m(m+n+1)}{2}$

pmf symmetric about $\frac{mn}{2}$

$$P(U_Y = u) = P\left(W_Y - \frac{m(m+1)}{2} = u\right) = P\left(W_Y = u + \frac{m(m+1)}{2}\right).$$

  - The tests based on U_Y and W_Y (or U_X and W_X) are actually equivalent.

Note 8 (A comparison of t-test and Mann-Whitney (M-W) test)

- Unlike t-test, the M-W test does not depend on normality assumption.

can be applied when F & G are not normal, e.g., Cauchy

Check Note 6 (LNp25) Note 7 (LNp.30)

⊙ The M-W test is insensitive to outliers, where as the t-test is sensitive.

based on ranks based on $\bar{X}, \bar{Y}$

⊙ When the normality assumption holds, the t-test is more powerful.

Check comparison (LNp.34)

- However, under normality assumption, the M-W test is nearly as powerful as the t-test. It has been shown that to attain the same power ← different sample sizes
  - the total sample size required for the t-test is approximately 0.95 times the total sample size required for the M-W test.

Check Note 6 (LNp25)

⊙ The M-W test is generally preferable, especially for small sample sizes.

Theorem 12 (means and variances of U_Y and W_Y under H_0)

Consider the nonparametric model ($\Diamond$) in LNp.35. If $\Delta = 0$ ($\Leftrightarrow \pi_\Delta = 1/2$),

Ho

Their pmfs are symmetric about their means

- $E(W_Y) = [m(m+n+1)]/2$ and $Var(W_Y) = [mn(m+n+1)]/12$
  ($\Leftrightarrow E(W_X) = [n(m+n+1)]/2$ and $Var(W_X) = [mn(m+n+1)]/12$ since $W_X = [(m+n)(m+n+1)]/2 - W_Y$)
- $E(U_Y) = mn/2$ and $Var(U_Y) = [mn(m+n+1)]/12$
  ($\Leftrightarrow E(U_X) = mn/2$ and $Var(U_X) = [mn(m+n+1)]/12$ since $U_X = mn - U_Y$)

same variance

$\because U_Y = W_Y - \frac{m(m+1)}{2}$

Note. $\because \hat{\pi}_\Delta = U_Y/mn$
Under H_0, $E(\hat{\pi}_\Delta) = 1/2$,
$Var(\hat{\pi}_\Delta) = \frac{m+n+1}{12\,mn} \to 0$
as $m, n \to \infty$

Proof. It is enough to prove the case of W_Y.

- Note that LNp.32 $W_Y \overset{d}{=} R_{n+1} + \cdots + R_{m+n}$.

  Under $H_0 : \Delta = 0$, $(R_{n+1}, \cdots, R_{m+n})$ can be viewed as a without-replacement simple random sample from the population

  check LNp.32

  $$\{1, \ldots, n, n+1, \ldots, m+n\}.$$

  N

  1 2 3 ... $m+n$

- Let $N = m + n$. Since

  population size

  $$\sum_{k=1}^{N} k = \frac{N(N+1)}{2} \quad \text{and} \quad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1))}{12},$$

  the population mean μ and variance σ^2 of this population distribution are

  $$\mu = \frac{1}{N}\left(\sum_{k=1}^{N} k\right) = \frac{N+1}{2} \quad \text{and} \quad \sigma^2 = \frac{1}{N}\left(\sum_{k=1}^{N} k^2\right) - \mu^2 = \frac{N^2-1}{12}.$$

- Let $\bar{R}$ ($= W_Y/m$) be the average of this without-replacement sample $(R_{n+1}, \ldots, R_{m+n})$. Then (by Thms 1 & 3 in LN, Ch7, p.16-18),

  sample mean

  sample size = m

  $$\underbrace{E(\bar{R}) = \mu}_{(N+1)/2} \quad \text{and} \quad Var(\bar{R}) = (\sigma^2/m)[(N-m)/(N-1)]. = \frac{n(n+m+1)}{12m}$$

  $\sigma^2 = (N-1)(N+1)/12$, $N - m = n$, finite population correction

- The results follows from $E(W_Y) = m\,E(\bar{R})$ and $Var(W_Y) = m^2\,Var(\bar{R})$.

Theorem 13 (Asymptotic null distribution of U_Y)

Consider the nonparametric model ($\Diamond$) in LNp.35 and the null $H_0 : \Delta = 0$ ($\Leftrightarrow \pi_\Delta = 1/2$). For m, n both greater than *10, the null distribution of U_Y (or U_X) is well approximated by a normal distribution, i.e.,

$$\frac{U_Y - E(U_Y)}{\sqrt{Var(U_Y)}} \overset{D}{\approx} N(0,1) \quad \left(\text{or} \quad \frac{U_X - E(U_X)}{\sqrt{Var(U_X)}} \overset{D}{\approx} N(0,1)\right)$$

$0 \leftarrow \frac{\hat{\pi}_\Delta - 1/2}{se(\hat{\pi}_\Delta)} \overset{D}{\approx} N(0,1)$

can be used to determine critical value & rejection region

The proof is omitted, but some **notes** are given below.

- This Thm does not follow immediatedly from the ordinary CLT although

  $$U_Y = \sum_i \sum_j Z_{ij} \quad \text{and} \quad Z_{ij} \sim \text{binomial}(1, \pi_\Delta).$$

  i.i.d. case

  But, Z_{ij}'s are not independent. check LNp.37 ~~i~~.i.d.?

- Similarly, the null distribution of W_Y (or W_X) can be approximated by normal, i.e.,

  $$\frac{W_Y - E(W_Y)}{\sqrt{Var(W_Y)}} \overset{D}{\approx} N(0,1) \quad \left(\text{or} \quad \frac{W_X - E(W_X)}{\sqrt{Var(W_X)}} \overset{D}{\approx} N(0,1)\right).$$

  $\because W_Y, W_X$ are linear transformation of U_Y, U_X, respectively

Example 5 (Asymptotic null dist. of W_Y, heat of fusion of ice, cont. Ex.4 in LNp.34)

- $n = 13$ (method A), $m = 8$ (method B), $W_B = 51$.
- Under the null, $\mu_{W_B} = E(W_B) = [8(8+13+1)]/2 = 88$,
  $\sigma_{W_B} = \sqrt{Var(W_B)} = \sqrt{[(8\times 13)(8+13+1)]/12} = 13.8$.
  known constant, not parameter
- Because $N(0,1) \overset{D}{\approx}$ r.v. $\dfrac{W_B - \mu_{W_B}}{\sigma_{W_B}}$ known constant, known constant $= \dfrac{51-88}{13.8} = -2.68$,

  the approximate p-value is $P(|N(0,1)| > 2.68) = 2\times[1-\Phi(2.68)] = 0.0074$
  ($\Rightarrow$ reject H_0 at $\alpha = 0.01 \Rightarrow$ consistent with the testing result using exact null distribution in Ex.4) check LNp.33

Theorem 14 (Nonparametric confidence interval for Δ)

Consider the nonparametric model $(\Diamond)$ in LNp.35.

duality between C.I. & testing

- **Q**: How can we test $H_0^*: \Delta = \Delta_0$ vs. $H_A^*: \Delta \neq \Delta_0$,

  where Δ_0 is a known constant?
  - Under H_0^*, we have (1) $X_i \sim F$, (2) $Y_j \sim G$, and (3) $G(x) = F(x-\Delta_0)$. unknown, known

    Then, new data $X_1^*, X_n^*, Y_1^*, Y_m^*$:
    $X_1, \ldots, X_n, Y_1 - \Delta_0, \ldots, Y_m - \Delta_0 \sim$ i.i.d. F ← Test LNp.32 $H_0: \Delta = 0$, $H_A: \Delta \neq 0$
  - The test of H_0^* vs. H_A^* using the data X_i's and Y_j's is equivalent to testing $H_0: \Delta = 0$ vs. $H_A: \Delta \neq 0$ using the data X_i's and $(Y_j - \Delta_0)$'s.
  - To test $H_0^*: \Delta = \Delta_0$, can use

1/8

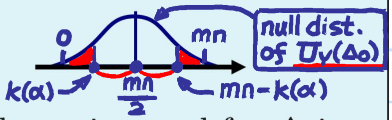
- $*$ the test statistic: $U_Y(\Delta_0) = \#\{X_i < Y_j - \Delta_0\} = \#\{Y_j - X_i > \Delta_0\}$, fixed, X_i^*, Y_j^*, changed
- $*$ the acceptance region: $k(\alpha) \le U_Y(\Delta_0) \le mn - k(\alpha)$, What if $\Delta_0 = 0$?

  Assume it is an integer

  where $k(\alpha)$ is the critical value determined by the significance level α

  irrelevant to F

  (**Note**. The null distribution of $U_Y(\Delta_0)$ is symmetric about $mn/2$.) check Thm10 (LNp.40)



- By the duality of test and C.I., a $100(1-\alpha)\%$ confidence interval for Δ is

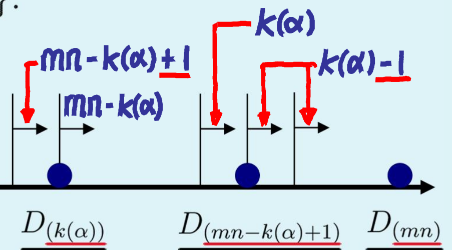
$$C = \{\Delta \mid k(\alpha) \le U_Y(\Delta) \le mn - k(\alpha)\}.$$

changed, fixed

$k(\alpha)$ can be obtained from the critical value of W_Y (check Thm11, LNp.39) in Table8, TBp A21

- Let $D_{(1)}, D_{(2)}, \ldots, D_{(mn)}$ denote the ordered mn differences $(Y_j - X_i)$'s. pivotal quantity

  Then, $C = [D_{(k(\alpha))},\ D_{(mn-k(\alpha)+1)}]$. What is its dist.?



- To see this,
  - $*$ if $\Delta_0 = D_{(k(\alpha))}$, then $U_Y(\Delta_0) = \#\{Y_j - X_i > \Delta_0\} = mn - k(\alpha)$, accept

    if $\Delta_0 < D_{(k(\alpha))}$, then $U_Y(\Delta_0) = \#\{Y_j - X_i > \Delta_0\} \ge mn - k(\alpha) + 1$, reject

    thus, $D_{(k(\alpha))}$ is the leftmost point of the confidence interval C,
  - $*$ if $\Delta_0 \le D_{(mn-k(\alpha)+1)}$, then $U_Y(\Delta_0) = \#\{Y_j - X_i > \Delta_0\} \ge k(\alpha)$, accept

    if $\Delta_0 \ge D_{(mn-k(\alpha)+1)}$, then $U_Y(\Delta_0) = \#\{Y_j - X_i > \Delta_0\} \le k(\alpha) - 1$, reject

    thus, $D_{(mn-k(\alpha)+1)}$ is the rightmost point of the confidence interval C.