Note 3 (Some notes about z- and t-tests)

c.f. 2-sided tests in LNp.13

- For the null and alternative hypotheses:

$$H_0: \Delta = \Delta_0 \ (\text{or } \Delta \leqslant \Delta_0) \ \text{ vs. } \ H_A^*: \Delta > \Delta_0 \quad \text{(need domain knowledge)}$$

or

$$H_0: \Delta = \Delta_0 \ (\text{or } \Delta \geqslant \Delta_0) \ \text{ vs. } \ H_A^{**}: \Delta < \Delta_0 \quad \text{(need domain knowledge)}$$

where H_A^* and H_A^{**} are **one-sided alternatives**, the z- and t-tests are

– σ^2 known: $Z \geq z(\alpha)$ for H_A^*, and $Z \leq -z(\alpha)$ for H_A^{**} ← (reasonable?)

– σ^2 unknown: $T \geq t_{m+n-2}(\alpha)$ for H_A^*, and $T \leq -t_{m+n-2}(\alpha)$ for H_A^{**}

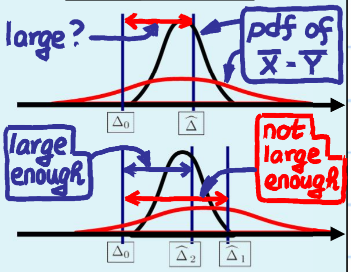
Q: Are these tests good tests?

- **FYI.** All the tests presented in LNp.13-14 are uniformly most powerful unbiased (UMPU) tests. (**Note.** Its proof follows a theorem of UMPU tests for exponential family with *nuisance parameters*)

$\mu_X - \mu_Y$

$\mu_X + \mu_Y$, σ^2

significanly large?

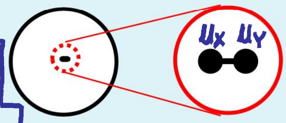


- The test statistics are of the form:

$$\frac{(\bar{X}-\bar{Y}) - \Delta_0}{s_{\bar{X}-\bar{Y}}} \ \text{ or } \ \sigma_{\bar{X}-\bar{Y}}.$$

a measure of the accuracy of the est'or $\bar{X}-\bar{Y}$

(estimated) std. error of $\bar{X}-\bar{Y}$ (Note 2, LNp.12)



– In the numerator, $(\bar{X}-\bar{Y}) - \Delta_0$ estimates $\Delta - \Delta_0$.

– Q: why is this estimate divided by $s_{\bar{X}-\bar{Y}}$ ($s_{\bar{X}-\bar{Y}} \downarrow$ when $m \uparrow$ and/or $n \uparrow$)?

functions as the scale of a ruler

- Q: if H_0 not rejected, do we really accept $\Delta = \Delta_0$, say $\mu_X = \mu_Y$? (better to claim "sample size is not large enough to reject H_0.")

$\Delta_0 = 0$

- statistically significant difference vs. physically significant difference (example?)

very easy to claim significant when sample size is large

statistical standard $\xleftarrow{\text{different}}\!\!\rightarrow$ physical standard

Theorem 6 (likelihood ratio tests for Δ=Δ_0, 2-sample normal model)

All the tests presented in LNp.13-14 are likelihood ratio tests.

Check Textbook Sec.9.4

Proof: We only prove the case of two-sided hypothesis. For the case of one-sided hypothesis, its proof is similar (exercise).

H_A

- Recall that

– the log-likelihood is

3 parameters: μ_X, μ_Y, σ^2

Thm2 (LNp.9)

$$l = \log(\mathcal{L}) \propto -\frac{m+n}{2}\log(\sigma^2) - \frac{\sum_{i=1}^n (X_i - \mu_X)^2}{2\sigma^2} - \frac{\sum_{j=1}^m (Y_j - \mu_Y)^2}{2\sigma^2},$$

likelihood

– the test statistic of likelihood ratio test is → used to determine what data is "more extreme", support H_A

an increasing function

Substitute MLEs

$$\Lambda = \frac{\sup_\omega \mathcal{L}}{\sup_\Omega \mathcal{L}} \ \text{ or } \ \log(\Lambda) = \sup_\omega \log(\mathcal{L}) - \sup_\Omega \log(\mathcal{L}) = \sup_\omega l - \sup_\Omega l,$$

≤ 1

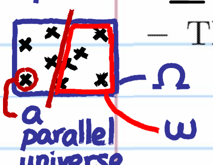
where $\Omega = H_0 \bigcup H_A$ and $\omega = H_0$, ← $\Delta = \Delta_0$

$\Delta \in \mathbb{R}$

– a likelihood ratio test rejects H_0 for small values of Λ (or $\log \Lambda$).

Why are they more extreme?

parameter space



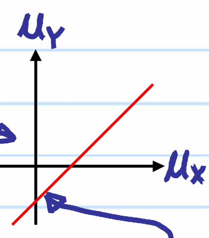
- σ^2 known

– The parameter spaces Ω and ω are

$$\Omega = \{(\mu_X, \mu_Y) \mid \mu_X \in \mathbb{R}, \ \mu_Y \in \mathbb{R}\}$$

$$\omega = \{(\mu_X, \mu_Y) \mid \mu_X \in \mathbb{R}, \ \mu_Y = \mu_X - \Delta_0\}$$

$\Omega (= \mathbb{R}^2)$



$\Delta = \Delta_0$ (H_0)

ω ($\mu_X - \mu_Y = \Delta_0$)

– Under Ω, the MLE's of (μ_X, μ_Y) are

Thm3 (LNp.9)

$$\hat{\mu}_{X,\Omega} = \bar{X}, \qquad \hat{\mu}_{Y,\Omega} = \bar{Y},$$

l in LNp.15 c.f.

$$\Rightarrow \sup_{\Omega} l = l(\hat\mu_{X,\Omega}, \hat\mu_{Y,\Omega}) \propto -\frac{\sum_{i=1}^n (X_i-\overline{X})^2 + \sum_{j=1}^m (Y_j-\overline{Y})^2}{2\sigma^2} - \frac{m+n}{2}\log(\sigma^2)$$

– Under ω, the log-likelihood is proportional to

$$l^*(\mu_X) \equiv -\frac{m+n}{2}\log(\sigma^2) - \frac{\sum_{i=1}^n (X_i-\mu_X)^2}{2\sigma^2} - \frac{\sum_{j=1}^m [Y_j - \underbrace{(\mu_X-\Delta_0)}_{=\mu_Y}]^2}{2\sigma^2},$$

σ^2: known constant

$X_1,\cdots,X_n$, $Y_1+\Delta_0,\cdots,Y_m+\Delta_0$ i.i.d. $N(\mu_X,\sigma^2)$

c.f. one-sample normal data (LNp.6) with $n+m$ observations

and the MLE's of (μ_X,μ_Y) are $0=\frac{dl^*}{d\mu_X} \propto 2\left[\sum_i X_i - n\mu_X + \sum_j (Y_j+\Delta_0) - m\mu_X\right]$

$\mu_X - \Delta_0 =$

$$\hat\mu_{X,\omega} = \frac{n}{m+n}\overline{X} + \frac{m}{m+n}(\overline{Y}+\Delta_0),$$

$$\hat\mu_{Y,\omega} = \hat\mu_{X,\omega} - \Delta_0 = \frac{n}{m+n}(\overline{X}-\Delta_0) + \frac{m}{m+n}\overline{Y},$$

$-\frac{m}{m+n}\Delta_0 - \frac{n}{m+n}\Delta_0$

by invariance property of MLE

$$\Rightarrow \sup_{\omega} l = l(\hat\mu_{X,\omega}, \hat\mu_{Y,\omega}) \propto -\frac{\sum_{i=1}^n (X_i-\hat\mu_{X,\omega})^2 + \sum_{j=1}^m (Y_j-\hat\mu_{Y,\omega})^2}{2\sigma^2} - \frac{m+n}{2}\log(\sigma^2)$$

– Therefore, the log-likelihood-ratio is

$\sup_\omega l - \sup_\Omega l$

$$\begin{aligned}
\log(\Lambda) &= l(\hat\mu_{X,\omega}, \hat\mu_{Y,\omega}) - l(\hat\mu_{X,\Omega}, \hat\mu_{Y,\Omega}) \\
&= -\frac{1}{2\sigma^2}\Big[\Big(\cancel{\textstyle\sum_{i=1}^n X_i^2} - 2n\overline{X}\hat\mu_{X,\omega} + n\hat\mu_{X,\omega}^2 + \cancel{\textstyle\sum_{j=1}^m Y_j^2} - 2m\overline{Y}\hat\mu_{Y,\omega} + m\hat\mu_{Y,\omega}^2\Big) \\
&\qquad + \Big(\cancel{\textstyle\sum_{i=1}^n X_i^2} + n\overline{X}^2 + \cancel{\textstyle\sum_{j=1}^m Y_j^2} + m\overline{Y}^2\Big)\Big] \\
&= -\frac{1}{2\sigma^2}\left[n(\overline{X}-\hat\mu_{X,\omega})^2 + m(\overline{Y}-\hat\mu_{Y,\omega})^2\right] \\
&= -\frac{1}{2\sigma^2}\left(\frac{mn}{m+n}\right)(\overline{X}-\overline{Y}-\Delta_0)^2
\end{aligned}$$

$\frac{m^2 n}{(m+n)^2} + \frac{n^2 m}{(m+n)^2}$

$$\overline{X}-\hat\mu_{X,\omega} = \frac{m}{m+n}(\overline{X}-\overline{Y}-\Delta_0)$$
$$\overline{Y}-\hat\mu_{Y,\omega} = \frac{-n}{m+n}(\overline{X}-\overline{Y}-\Delta_0)$$

– The likelihood ratio test rejects H_0 for $\Delta=\Delta_0$

small values of $\log(\Lambda)$ $\Leftrightarrow$ large values of $|(\overline{X}-\overline{Y})-\Delta_0|$,

10/25

which is the z-test apart from constants that do not depend on the data.

Thm5 (LNp.13)

- σ^2 unknown

parameter

– The parameter spaces Ω and ω are

Ω, ω in LNp.15 c.f.

$$\Omega = \{(\mu_X,\mu_Y,\sigma^2) \mid \mu_X\in\mathbb{R},\ \mu_Y\in\mathbb{R},\ \sigma^2>0\}$$
$$\omega = \{(\mu_X,\mu_Y,\sigma^2) \mid \mu_X\in\mathbb{R},\ \mu_Y=\mu_X-\Delta_0,\ \sigma^2>0\}$$

$\Delta=\Delta_0$ (H_0)

– Under Ω, the MLE's of (μ_X,μ_Y,σ^2) are

Thm3 (LNp.9~10)

$$\hat\mu_{X,\Omega} = \overline{X}, \qquad \hat\mu_{Y,\Omega} = \overline{Y},$$

$$\hat\sigma_\Omega^2 = \frac{1}{m+n}\Big[\sum_{i=1}^n (X_i-\underbrace{\hat\mu_{X,\Omega}}_{\overline{X}})^2 + \sum_{j=1}^m (Y_j-\underbrace{\hat\mu_{Y,\Omega}}_{\overline{Y}})^2\Big] = \frac{(n-1)S_X^2+(m-1)S_Y^2}{m+n}$$

$m+n-2$ c.f. $m+n$

$\frac{m+n-2}{m+n} S_p^2$

c.f.

$$\Rightarrow l(\hat\mu_{X,\Omega}, \hat\mu_{Y,\Omega}, \hat\sigma_\Omega^2) \propto -\frac{m+n}{2}\log(\hat\sigma_\Omega^2) - \frac{\sum_{i=1}^n (X_i-\overline{X})^2 + \sum_{j=1}^m (Y_j-\overline{Y})^2}{2\hat\sigma_\Omega^2}$$

$\sup_\Omega l =$

l in LNp.15 c.f.

$(m+n)\hat\sigma_\Omega^2$

$$= -\frac{m+n}{2}\log(\hat\sigma_\Omega^2) - \frac{m+n}{2}$$

– Under ω, the log-likelihood is proportional to

$X_1,\cdots,X_n,Y_1+\Delta_0,\cdots,Y_m+\Delta_0$ i.i.d. $N(\mu_X,\sigma^2)$

$$l^* \equiv -\frac{m+n}{2}\log(\sigma^2) - \frac{\sum_{i=1}^n (X_i-\mu_X)^2}{2\sigma^2} - \frac{\sum_{j=1}^m [Y_j - \underbrace{(\mu_X-\Delta_0)}_{=\mu_Y}]^2}{2\sigma^2}$$

σ^2: parameter

and the MLE's of (μ_X,μ_Y,σ^2) are ← solve $\partial l^*/\partial\mu_X=0$ and $\partial l^*/\partial\sigma^2=0$

$$\hat\mu_{X,\omega} = \frac{n}{m+n}\overline{X} + \frac{m}{m+n}(\overline{Y}+\Delta_0),$$ ← same as σ^2 known (LNp.16)