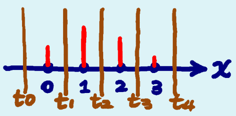
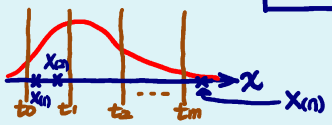
Remarks.

1. There is a distinction between $O_1,\dots,O_m$ and $X_1,\dots,X_n$ (especially for continuous case) empirical cdf (textbook, sec. 10.2) ⇕ → converge to true cdf as $n\to\infty$
   - $X_1,\dots,X_n \Rightarrow X_{(1)},\dots,X_{(n)} \Rightarrow O_1,\dots,O_m$
   - $O_1,\dots,O_m \underset{\text{continuous}}{\overset{\text{discrete (possible)}}{\underset{\cancel{\Rightarrow}}{\Rightarrow}}} X_{(1)},\dots,X_{(n)} \underset{\text{if not i.i.d.}}{\overset{\text{if i.i.d.}}{\underset{\cancel{\Rightarrow}}{\Rightarrow}}} X_1,\dots,X_n$

If $X_1,\cdots,X_n$ are i.i.d., order statistics $X_{(1)},\cdots,X_{(n)}$ are sufficient for any distribution. $f_{X_1,\cdots,X_n}(x_1,\cdots,x_n)/f_{X_{(1)},\cdots,X_{(n)}}(x_{(1)},\cdots,x_{(n)})=\frac{1}{n!}$

$X_1,\cdots,X_n$: discrete r.v.'s





$X_1,\cdots,X_n$: continuous r.v.'s

2. The MLE of θ based on $O_1,\dots,O_m$ can be different from the MLE of θ based on $X_1,\dots,X_n$.
3. Different choices of $(t_{i-1},t_i]$, $i=1,\dots,m$, can cause different results. (**Note.** The choice should not depend heavily on observed data.)

Note. In Ex.7.17, t_i's are not functions of data, i.e., t_i's are not statistics, not r.v.'s, ~~$t_i(X_1,\cdots,X_n)$~~.

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4. It is recommended that O_i, $E_i \geq 5$. ← a result guaranteed by large n ← asymptotic property

Example 7.19 (Hardy-Weinberg Equilibrium, TBp.343-344, or Ex.6.15, LN, Ch8, p.3-24)

- $n=1029$, the cell probabilities are $\underset{\text{AA}}{(1-\theta)^2}$, $\underset{\text{Aa}}{2\theta(1-\theta)}$, $\underset{\text{aa}}{\theta^2}$ under the Hardy-Weinberg Equilibrium model and the MLE of θ is $\hat\theta=0.4247$.

$X_1,\cdots,X_{1029} \overset{\text{i.i.d.}}{\sim}$ multinomial$(1,p_1,p_2,p_3)$ → $\in\{1,2,3\}$

$X_{(1)},\cdots,X_{(1029)}$

$O_1,O_2,O_3\sim$ multinomial$(1029,p_1,p_2,p_3)$

	Blood Type		
	M	MN	N
O_i	342	500	187
E_i	340.6	502.8	185.6

→ intuitive conclusion?

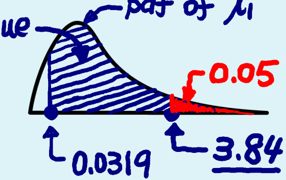
how to get this? e.g. $340.6=n\,p_1(\hat\theta)=1029\cdot(1-0.4247)^2$

- Consider the test:

Ω_0 ← H_0: $(p_1(\theta),p_2(\theta),p_3(\theta))$ are specified by the Hardy-Weihberg model

Ω ← H_A: (p_1,p_2,p_3) do not have that specified form → $\Omega: p_1+p_2+p_3=1$

- Pearson's chi-square test: $\boxed{\dim(\Omega_0)=1,\ \dim(\Omega)=2}$
  1. Pearson's chi-square test statistic is
$$X^2=\sum_{i=1}^{\overset{ⓜ}{3}}\frac{(O_i-E_i)^2}{E_i}\ \underset{\because n=1029}{\xrightarrow{D}}\ \underset{\dim(\Omega)-\dim(\Omega_0)}{\overset{\text{asymptotic null distribution}}{\chi_1^2}}\ \text{under } H_0.$$



meaning?

  2. Set $\alpha=0.05$. Thus, reject H_0 if the value of X^2 statistic exceeds 3.84, the 95%-quantile of the χ_1^2 distribution.

critical value

  3. Since
$$X^2=\frac{(\overset{1.4^2}{\overset{\parallel}{342-340.6}})^2}{340.6}+\frac{(500-502.8)^2}{502.8}+\frac{(\overset{1.4^2}{\overset{\parallel}{187-185.6}})^2}{185.6}\doteq 0.0319,$$
H_0 is not rejected.

2. Why $\alpha = 0.05$? Will conclusion be different if we choose other α?

The p-value is more useful:

$$p\text{-value} = P_{H_0}(\underbrace{X^2 > \overbrace{0.0319}^{X^2_{obs}}}_{\longleftarrow \text{more extreme}}) = P(\chi_1^2 \ge 0.0319) = 0.86.$$

H_0

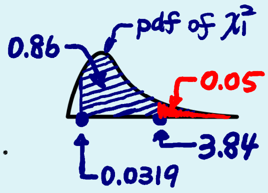
If the null model were correct, deviations this large or larger would occur 86% of the time. Thus, the data give us no reason to doubt the null model.

as extreme or more extreme

- Likelihood ratio statistic is

$$-2\log\Lambda = 2\textstyle\sum_{i=1}^{3} O_i \log\left(\frac{O_i}{E_i}\right) = 0.032 \quad (\approx 0.0319 = X^2).$$

Why? LNp.42



The two tests leads to the same conclusion. meaning? ← check Λ in LNp.32

Note: $\Lambda = \exp(0.032/(-2)) = 0.98 \approx 1$ $(0 \le \Lambda \le 1)$. Hardy-Weinberg model is almost as likely as the most general possible model.

Example 7.20 (Bacterial Clumps, TBp. 344-345)

- In testing milk for bacterial contamination, 0.01mL of milk is spread over an area of 1cm². 400 counts of bacterial clumps:

sample size

$X_1, \cdots, X_{400}$

number per 1cm²	0	1	2	⋯	9	10	19
Frequency	56	104	80	⋯	3	2	1

histogram

$X_1, \cdots, X_{400}$ i.i.d. $P(\lambda)$

$O_1, \cdots, O_m$

possible statistical modeling
for $X_1, \cdots, X_{400}$?
for $O_1, \cdots, O_m$?

- H_0: The data are from Poisson $P(\lambda)$

- MLE for the λ of Poisson model (H_0) is ← **Note.** The MLE $\hat{\lambda}$ is the MLE based on $X_1, \cdots, X_{400}$, not the MLE based on $O_1, \cdots, O_m$. But, in the case, their MLEs are identical.

$$\overline{X} = \hat{\lambda} = \frac{0 \times 56 + 1 \times 104 + \cdots + 19 \times 1}{400} = 2.44,$$

giving the expected frequencies E_i in the following table.

number per 1cm²	0	1	2	⋯	6 Why?	≥ 7
O_i (Obs. freq.)	56	104	80	⋯	9	20
E_i (Exp. freq.)	34.9	85.1	103.8	⋯	10.2	5.0
Component of χ^2	12.8	4.2	5.5	⋯	0.14	45.0

23.8

15

$\frac{(O_i-E_i)^2}{E_i}$

how to get it?

$O_1, \cdots, O_8 \sim$ multinomial$(400, p_1, \cdots, p_8)$
H_0: $p_1(\lambda), \cdots, p_8(\lambda)$

critical value

- The chi-square statistic is $X^2 = 75.6 > 18.55 = \chi_6^2(0.005)$. So, p-value < 0.005 and the goodness of fit test rejects the Poisson model (H_0).

2.887×10^{-14}

$\dim(\Omega) = 7$, $\dim(\Omega_0) = 1$

Why? How assumptions in Poisson violated?

1. bacteria held by surface tension on lower surface of the drop may adhere to the glass slide on contact.
2. film not of uniform thickness.

New statistical modeling:
Poisson → Negative binomial
(LN. CH8. p.3-68)

Example 7.21 (Fisher's reexamination of Mendel's data, TBp. 345-346)

- Mendel crossed 556 smooth, yellow male peas with wrinkled, green female peas.

n

AA Aa aA

BB Bb bB

aa

bb

$\{1,2,3,4\} \ni X_1, \cdots, X_{556} \overset{\text{i.i.d.}}{\sim} \text{multinomial}(1, p_1, p_2, p_3, p_4)$

	Type	Observed Count	Expected Count	Probability
$\{AA, Aa, aA\} \times \{BB, Bb, bB\}$	Smooth yellow	O_1 315	E_1 312.75	p_{10} 9/16
$\{AA, Aa, aA\} \times \{bb\}$	Smooth green	O_2 108	E_2 104.25	p_{20} 3/16
$\{aa\} \times \{BB, Bb, bB\}$	Wrinkled yellow	O_3 102	E_3 104.25	p_{30} 3/16
$\{aa\} \times \{bb\}$	Wrinkled green	O_4 31	E_4 34.75	p_{40} 1/16

$O_1, \cdots, O_4 \sim \text{multinomial}(\underline{556}, p_1, \cdots, p_4)$

How to get them?

- For the data, $\underline{\Omega_0}: p_1 = \frac{9}{16}, p_2 = \frac{3}{16}, p_3 = \frac{3}{16}, p_4 = \frac{1}{16}$, $\underline{\Omega} = \{(p_1, \cdots, p_4) \mid p_1 + p_2 + p_3 + p_4 = 1\}$

  $\underline{-2\log\Lambda} = 2\sum_{i=1}^{4} \underline{O_i} \log(\underline{O_i}/\underline{E_i}) = \underline{0.618}$, $\dim(\Omega) = \underline{3}$, $\dim(\Omega_0) = \underline{0}$

  $\underline{\Lambda} = \exp(-0.618/2) = \underline{0.73}$, the $\underline{p\text{-value}}$ is slightly less than $\underline{0.9}$. (Peason's statistic is $\underline{X^2 = 0.604}$.) ← asymptotic null distribution: $\chi^2_{\underline{3}}$ $(n=556)$

- Fisher pooled the results of all of Mendel's experiments:
  - Two independent experiments give chi-square statistic T_1, T_2 with p and r degrees of freedom under H_0. indep. $n_1 = 556$ n_2 from different datasets
  - Under H_0, $T_1 + T_2 \sim \chi^2_{p+r}$. ← $p = r = 3$ in the case.
  - Adding all the chi-square statistics for all the independent experiments gives p-value=0.99996! ← too good to be true
- The best explanation is perhaps that Mendel continued experimenting until the results looked good. The statistical analysis here assume n is fixed before data are collected. Then, $\underline{n}$ is a random variable

- Dorfman (1978) studied the goodness of fit of the intelligence scores of fathers and sons to a normal distribution (H_0) using Pearson's chi-square test. The p-values were greater than $1 - 10^{-7}$ and $1 - 10^{-6}$, respectively.

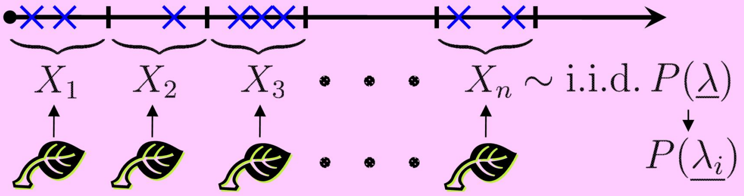
→ too good to be true?

❖ **Reading**: textbook, 9.5

- Application of GLR test II --- Poisson dispersion test

Question 7.16

- Recall the insect counts example (Ex. 6.31, LN, Ch8, p.68), the Poisson model did not fit well.
- Poisson model assumptions:
  1. The rate is constant. ← 次 / 單位時間
  2. Counts in one interval are independent of counts in disjoint intervals.



- For counts of insects on leaves, some assumptions may be violated, e.g.,
  - Leaves are of different sizes and occur at various locations on different plants. Hence, 1. may fail.
  - If the insects hatched from eggs that were deposited in groups, there may be clustering of the insects. Then, 2. may fail.
- How to examine whether the rate is a constant for the insect data? → But, still assume $X_1, \cdots, X_n$ are independent Poisson.

Example 7.22 (GLR test for Poisson dispersion, TBp. 347-348)

cf. Ex.7-20 (LNp.45)

- statistical modeling for solving the question:
  - $X_i \sim P(\lambda_i), i = 1, \dots, n$. & $X_1,\cdots,X_n$ are independet. → joint pmf $\prod_{i=1}^{n} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}$

$$\Omega = \{(\lambda_1, \dots, \lambda_n) : \lambda_i > 0, i = 1, \dots, n\} \qquad \Rightarrow \dim(\Omega) = n.$$

  - Null hypothesis H_0: the counts are Poisson with common parameter λ.

$X_1,\cdots,X_n$ i.i.d. $P(\lambda)$ ←

$$\Omega_0 = \{(\lambda_1, \dots, \lambda_n) : \lambda_1 = \cdots = \lambda_n \equiv \lambda\} \qquad \Rightarrow \dim(\Omega_0) = 1.$$

  - Alternative hypothesis H_A: the counts are Poisson with different rates $\lambda_1, \lambda_2, \dots, \lambda_n$, i.e., $\Omega_A = \Omega \backslash \Omega_0$.

- Under Ω_0, MLE of λ is $\hat{\lambda} = \overline{X}$.
- Under Ω, MLE of λ_i's are X_i's, denoted by $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n$.
- Thus the likelihood ratio is

$O_i \leftrightarrow X_i$
$E_i \leftrightarrow \overline{X}$

test statistic in Ex.7.17 (LNp.40) cf.

$$\Lambda = \frac{\prod_{i=1}^{n} \hat{\lambda}^{x_i} e^{-\hat{\lambda}} / \cancel{x_i!}}{\prod_{i=1}^{n} \tilde{\lambda}_i^{x_i} e^{-\tilde{\lambda}_i} / \cancel{x_i!}} = \prod_{i=1}^{n} \left(\frac{\overline{x}}{x_i}\right)^{x_i} e^{x_i - \overline{x}}$$

test statistic in Ex.7.18 (LNp.41)

$$-2\log\Lambda = 2\sum_{i=1}^{n}\left[x_i \log\left(\frac{x_i}{\overline{x}}\right) + (x_i - \overline{x})\right] = 2\sum_{i=1}^{n} x_i \log\left(\frac{x_i}{\overline{x}}\right)$$

cf. check the proof in LNp.42 → Under H_0

$$\approx \frac{1}{\overline{x}}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{n\,\hat{\sigma}^2}{\overline{x}}$$

← (reasonable?) $\hat{\sigma}^2$: sample variance

dispersion index, if >> 1

  - $\hat{\sigma}^2/\overline{x}$: measure of clustering ← check the graph in LNp.48
  - Null Poisson model: variance=mean.
  - Alternative Poisson model: variance>mean.
  - (asymptotic) null distribution: χ^2_{n-1} ← $\dim(\Omega) - \dim(\Omega_0)$ check LNp.39

$X_i|\lambda_i \sim P(\lambda_i)$
$\lambda_i \geq 0$: r.v.
$E(\lambda_i) > 0$
⇓
$\frac{Var(X_i)}{E(X_i)} = [Var(\lambda_i) + E(\lambda_i)]/E(\lambda_i)$

- $Y \sim P(\lambda) \Rightarrow E(Y) = \lambda, Var(Y) = \lambda$.
- If $Y_1,\cdots,Y_n$ i.i.d. $P(\lambda)$

$$\overline{Y} = \widehat{E(Y_i)}$$
$$\approx$$
$$\frac{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}{n} = \widehat{Var(Y_i)}$$

- Poisson dispersion test (for goodness-of-fit)
  - has high power against alternative that are overdispersed relative to Poisson, ← e.g. $X_1,\cdots,X_n$ i.i.d. negative binomial (LN, CH8, p.65~66)

    i.e. $Var(Y) >> E(Y)$
  - often used when there is not enough data to be accumulated into several cells. ← $X_1,\cdots,X_n \rightarrow O_1,\cdots,O_m$ (better to have $O_i \geq 5$)

use original data $X_1,\cdots,X_n$, not $O_1,\cdots,O_m$

Example 7.23 (Asbestos Fibers, Poisson dispersion test, TBp. 348)

- For the data in Ex. 6.4, LN, Ch8, p.9, $n = 23$. ← not large

$n\hat{\sigma}^2/\overline{x} = \frac{1}{\bar{x}}\sum_{i=1}^{n}(x_i - \bar{x})^2 = 26.56$; $-2\log\Lambda = 2\sum_{i=1}^{n} x_i \log\left(\frac{x_i}{\bar{x}}\right) = 27.11$

- $\dim(\Omega) - \dim(\Omega_0) = 23 - 1 = 22$ → asymptotic null dist: χ^2_{22} ← questionable
- p-value for 27.11 is 0.21. So there is not enough evidence against the null hypothesis.
- **Note.** Sample size 23 is small and the test may have low power.

Example 7.24 (Bacterial Clumps, Poisson dispersion test, TBp. 348-349)

- For the data in Ex.7.20, LNp.45, n=400.

$$\overline{x} = 2.44, \quad \hat{\sigma}^2 = 4.59 \quad \Rightarrow \quad \frac{n\hat{\sigma}^2}{\bar{x}} = \frac{400 \times 4.59}{2.44} = 752.7$$

- The p-value is:

$$p\text{-value} = P\left(\frac{n\hat{\sigma}^2}{\overline{X}} \geq 752.7 \,\Big|\, H_0\right) = P\left(\frac{\frac{n\hat{\sigma}^2}{\overline{X}} - 399}{\sqrt{2 \times 399}} \geq \frac{752.7 - 399}{\sqrt{2 \times 399}}\right)$$

D → N(0,1)

compare it with the p-value in Ex.7.20 (LNp.46)

$$\approx 1 - \Phi(12.5) \approx 0 \quad \text{(normal approximation to } \chi^2_{m=399})$$

↳ 3.733×10^{-36}

$\dim(\Omega) = 400$, $\dim(\Omega_0) = 1$

- Thus, there is almost no doubt that the Poisson distribution fails to fit the data.

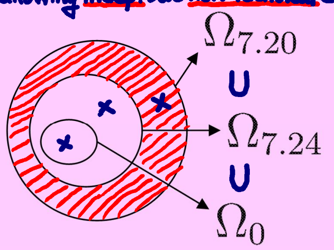
$\chi^2_m \sim Y = \underset{\chi_1^2}{Y_1} + \cdots + \underset{\chi_1^2}{Y_m} \xrightarrow{CLT}$ Normal

Question 7.17

- Compare Ex. 7.20 and Ex. 7.24. They test the same null hypothesis H_0. Why are the test statistics in the two examples different? (check Q in LNp.38)

$\boxed{X_1, \cdots, X_{400} \text{ i.i.d. } P(\lambda)}$

  - In Ex. 7.20, ← goodness-of-fit test.
    $\Omega_{7.20} = \{X_i$ can be any discrete r.v.'s$\}$ $\boxed{O_i \approx E_i?}$ allowing indep. but non-identical dist.
  - In Ex. 7.24, ← Poisson dispersion test
    $\Omega_{7.24} = \{X_i \sim \underset{\text{specified}}{P}(\lambda_i), i = 1, \ldots, n.\}$ $\boxed{\text{variance} \approx \text{mean?}}$



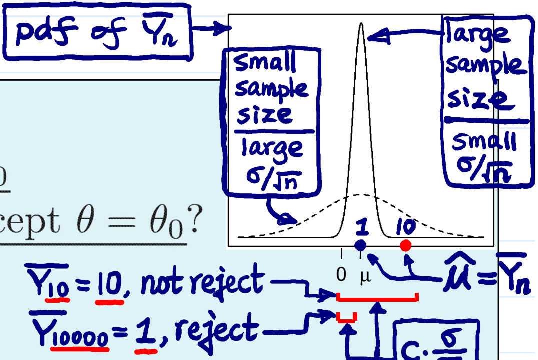
- Is it appropriate to use the test statistic in Ex. 7.20 to test the H_0 and H_A in Ex. 7.24? How is the opposite?
- For same data set, which of the tests in the two example would be expected to have smaller p-value? Why? ← (i) $\underset{\sim}{\theta} \in \Omega_{7.24} \setminus \Omega_0$ (ii) $\underset{\sim}{\theta} \in \Omega_{7.20} \setminus \Omega_{7.24}$

Note. If one has a specific alternative hypothesis in mind, better power can be obtained by developing a test against that alternative rather than against a more general alternative.

❖ **Reading**: textbook, 9.6

$\boxed{5/19}$

- Some concerns about hypothesis testing



- Question: Suppose modeling is correct. For
  $$H_0\colon \theta = \theta_0 \quad \text{vs.} \quad H_A\colon \theta \neq \theta_0$$
  when H_0 is not rejected, does it mean we accept $\theta = \theta_0$?

$\boxed{\begin{array}{l} H_0: \mu = 0 \\ H_A: \mu \neq 0 \end{array}}$

e.g.: $Y_1, \ldots, Y_n \sim N(\mu, \sigma^2)$, σ known, $\mu \approx 0$, but not zero, reject H_0 if

$\overline{Y_{10}} = 10$, not reject

$\overline{Y_{10000}} = 1$, reject

$\boxed{c \cdot \frac{\sigma}{\sqrt{n}}}$

based on N(0,1), irrelevant to n & σ

$$\left|\frac{\overline{Y} - 0}{\sigma/\sqrt{n}}\right| \geq c \Leftrightarrow |\overline{Y}| > c\,\frac{\sigma}{\sqrt{n}} = c\sqrt{Var(\overline{Y})}.$$

LNp.33~34

precision
$\frac{\sigma}{\sqrt{n}}$ ↓ when n ↑
$\frac{\sigma}{\sqrt{n}}$ ↓ when σ ↓

1st exp't: $\overline{Y} = 10$ not reject

Consider the two cases:
(i) n=10, and (ii) n=10000.

2nd exp't: $\overline{Y} = 1$ reject