- Is it appropriate to use the test statistic in Ex. 7.20 to test the H_0 and H_A in Ex. 7.24? How is the opposite?
- For same data set, which of the tests in the two example would be expected to have smaller p-value? Why? ← (i) $\theta \in \Omega_{7.24} \backslash \Omega_0$ (ii) $\theta \in \Omega_{7.20} \backslash \Omega_{7.24}$

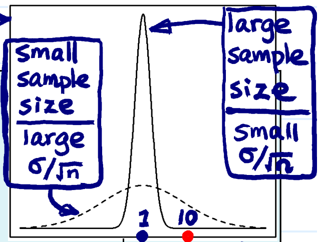
Note. If one has a specific alternative hypothesis in mind, better power can be obtained by developing a test against that alternative rather than against a more general alternative.

5/13

❖ **Reading**: textbook, 9.6

- Some concerns about hypothesis testing

pdf of $\bar{Y}_n$



- Question: Suppose modeling is correct. For
  H_0: $\theta = \theta_0$ vs. H_A: $\theta \neq \theta_0$
  when H_0 is not rejected, does it mean we accept $\theta = \theta_0$?

H_0: $\mu=0$
H_A: $\mu \neq 0$

e.g.: $Y_1, \ldots, Y_n \sim N(\mu, \sigma^2)$, σ known,
$\mu \approx 0$, but not zero, reject H_0 if

based on $N(0,1)$, irrelevant to n & σ

$$\left|\frac{\bar{Y}-0}{\sigma/\sqrt{n}}\right| \geq c \Leftrightarrow |\bar{Y}| > c\,\frac{\sigma}{\sqrt{n}} = c\sqrt{Var(\bar{Y})}.$$

LNp.33~34

$\bar{Y}_{10} = 10$, not reject
$\bar{Y}_{10000} = 1$, reject
$\hat{\mu} = \bar{Y}_n$
$c \cdot \frac{\sigma}{\sqrt{n}}$

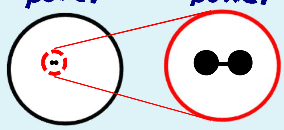
Consider the two cases:
(i) n=10, and (ii) n=10000.

1st exp't: $\bar{Y} = 10$ not reject

2nd exp't: $\bar{Y} = 1$ reject

precision
$\frac{\sigma}{\sqrt{n}} \downarrow$ when $n \uparrow$
$\frac{\sigma}{\sqrt{n}} \downarrow$ when $\sigma \downarrow$

low power   high power



Note: this is why we prefer **not** to say
"accept H_0",
but rather
"sample size is not large enough to reject H_0",
or "fail to reject H_0"
① H_0 is true
② information not enough

- When sample size n is large enough, it is very possible that almost every tests are significant (i.e., H_0 rejected).

  example: average temperature yesterday: $\mu_1 = 28°C$,
  average temperature today: $\mu_2 = 28.001°C$,
  $\mu_1 - \mu_2 = 0.001°C$.

  Test H_0 : $\mu_1 - \mu_2 = 0$ vs. H_A : $\mu_1 - \mu_2 \neq 0$. The null $\mu_1 = \mu_2$ will be easily rejected if n is very large. → μ_1 & μ_2 are (statistically) significantly different.
- Statistical significance may be inconsistent with practical/physical significance. (example?) ← Can you really "feel" the 0.001°C difference?
  - Q: what causes the inconsistency? To claim significance,

    precision of statistics → statistical standard $\xleftrightarrow{\text{different}}$ physical standard
  - for datasets with large n, easy to get *statistically significant* results for θ_i's (e.g., every $H_0 : \theta_i = 0$ rejected). But, the magnitudes of some θ_i's may be small (i.e., $\hat{\theta}_i \approx 0$) ⇒ not physically important.

• duality of confidence intervals and hypothesis tests

check Question 7.2 in LNp.2

Example 7.25 (Normal mean, TBp. 337)

- Let $X_1, \ldots, X_n$ be i.i.d. from $N(\mu, \sigma^2)$, where μ: unknown, σ^2: known.

parameter

- Null and alternative hypotheses:

$$H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_A : \mu \neq \mu_0$$

μ_0 can be set to be any value in $\mathbb{R}$

- Test statistic and rejection region: rejects H_0 for

$\bar{X}-\mu_0 \sim N(0, \frac{\sigma^2}{n})$ under H_0

$$|\bar{X} - \mu_0| > x_0 .$$

Check Ex.7.12 (LNp.27) Ex.7.15 (LNp.34)

- The x_0 is determined by

$\sim N(\mu_0, \frac{\sigma^2}{n})$ under H_0

$$P\left(|\bar{X} - \mu_0| > x_0 \middle| H_0\right) = \alpha \quad \Rightarrow \quad x_0 = \frac{\sigma}{\sqrt{n}} z(\alpha/2).$$

- Thus, the test accepts $H_0 : \mu = \mu_0$ when

a pivotal quantity if μ_0 regarded as a parameter, instead of a value

$$|\bar{X} - \mu_0| \underline{\underline{<}} \frac{\sigma}{\sqrt{n}} z(\alpha/2). \Leftarrow \text{acceptance region (AR)}$$

cf. Definition of confidence interval

$$\Leftrightarrow \quad P(AR \,|\, H_0) = 1 - \alpha.$$

only depend on data

true for any μ_0

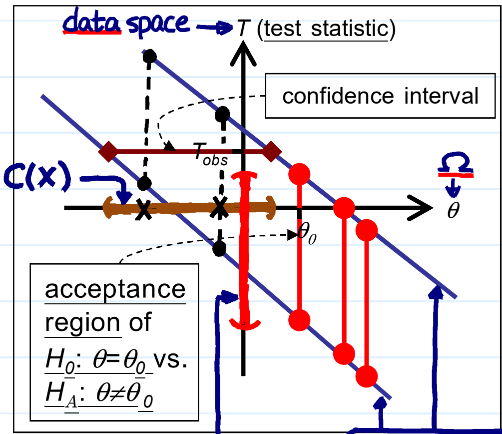
$$\Leftrightarrow \quad P\left(\bar{X} - \frac{\sigma}{\sqrt{n}} z(\alpha/2) < \mu_0 < \bar{X} + \frac{\sigma}{\sqrt{n}} z(\alpha/2) \middle| \mu = \mu_0\right) = 1 - \alpha.$$

- On the other hand, a $(1-\alpha) \times 100\%$ confidence interval for μ is

$$\left[\bar{X} - \frac{\sigma}{\sqrt{n}} z(\alpha/2),\ \bar{X} + \frac{\sigma}{\sqrt{n}} z(\alpha/2)\right]$$

- Thus, μ_0 lies in the $(1-\alpha) \times 100\%$ confidence interval if and only if the level-α test accepts $H_0 : \mu = \mu_0$.

Check item 3, LN, Ch8, p.81



AR(θ_0)

determined by a pivotal quantity

In general,

θ : parameter of a family of distributions

Ω : the set of all possible values of θ (parameter space)

Theorem 7.8 (TBp. 338)

Suppose for every θ_0 in Ω, there is a level-α test of the hypothesis

$$H_0 : \theta = \theta_0 \quad \text{vs.} \quad H_A : \theta \neq \theta_0$$

Denote the acceptance region of the test by $AR(\theta_0)$. Then the set

$$C(X) = \left\{\theta \,\middle|\, X \in AR(\theta)\right\}$$

defined confidence region in terms of acceptance region

is a $(1-\alpha) \times 100\%$ confidence region for θ.

Proof. First note that

$$P\left(X \in AR(\theta_0) \,\middle|\, \theta = \theta_0\right) \overset{\because \text{ level } \alpha \text{ test}}{=} 1-\alpha.$$

This implies that

Note. $\underset{\approx}{X} \in A(\theta_0) \Leftrightarrow C(\underset{\approx}{X}) \ni \theta_0$ random

$$\text{For any } \theta_0 \rightarrow P\left(\theta_0 \in C(X) \,\middle|\, \theta = \theta_0\right) = P\left(X \in AR(\theta_0) \,\middle|\, \theta = \theta_0\right) = 1-\alpha.$$

Theorem 7.9 (TBp. 338)

Suppose $C(X)$ is a $(1-\alpha)\times 100\%$ confidence region for θ. Then an acceptance region for a level-α test of $H_0: \theta = \theta_0$ is

$$AR(\theta_0) = \left\{X \,\middle|\, \theta_0 \in C(X)\right\}.$$

define acceptance region in terms of confidence region

Proof.

$$P(AR|H_0) \rightarrow P\left(X \in AR(\theta_0) \,\middle|\, \theta = \theta_0\right) = P\left(\theta_0 \in C(X) \,\middle|\, \theta = \theta_0\right) \overset{\text{by definition of C.I.}}{=} 1-\alpha.$$

Note.

- Theorem 7.8 says that a $(1-\alpha)\times 100\%$ confidence region for θ consists of all the values of θ_0 for which $H_0: \theta = \theta_0$ will not be rejected at level α.
- Theorem 7.9 says that the nlll hypothesis $H_0: \theta = \theta_0$ is accepted if θ_0 lies in the confidence region. $\Rightarrow$ can use confidence interval to do test

Example 7.26 (Normal variance)

- Let $X_1, \cdots, X_n$ be an i.i.d. sample from $N(\mu, \sigma^2)$, where μ is known and σ is unknown. (parameter)
- The acceptance region of the level-α UMP unbiased test for

$$H_0: \sigma = \sigma_0 \quad \text{vs.} \quad H_A: \sigma \neq \sigma_0$$

Ex7.13, LNp.29~30

is approximately

$$\chi_n^2(1-\alpha/2) \le \boxed{\frac{\sum_{i=1}^n (X_i-\mu)^2}{\sigma_0^2}} \le \chi_n^2(\alpha/2)$$

a pivotal quantity when σ_0 regarded as a parameter

equal tail probability

- It is equivalent to

$$\frac{\sum_{i=1}^n (X_i-\mu)^2}{\chi_n^2(\alpha/2)} \le \sigma_0^2 \le \frac{\sum_{i=1}^n (X_i-\mu)^2}{\chi_n^2(1-\alpha/2)}.$$

- Therefore, a $(1-\alpha)\times 100\%$ confidence interval for σ^2 is

$$\left[\frac{\sum_{i=1}^n (X_i-\mu)^2}{\chi_n^2(\alpha/2)}, \frac{\sum_{i=1}^n (X_i-\mu)^2}{\chi_n^2(1-\alpha/2)}\right].$$

Check item 5 LN, CH8, p.82

- a good test $\Leftrightarrow$ a good confidence interval (region)
- UMPU test $\Leftrightarrow$ UMA (Uniformally Most Accurate) confidence interval

❖ **Reading**: textbook, 9.3