Question 7.10

What if we are interested in $H_0: \theta \geq \theta_0$ vs. $H_A: \theta < \theta_0$?

$H_A: \theta < \theta_0$ smaller T more extreme

$$\theta^* = -\theta$$
$$C^*(\theta^*) = -C(-\theta^*)$$
$$T^*(\mathbf{x}) = -T(\mathbf{x})$$

$C^* T^* = C T$

C^*: an increasing function of θ^*

$H_0: \theta^* \leq -\theta_0$ vs. $H_A: \theta^* > -\theta_0$ | $C^*(\theta^*) \uparrow$, as $\theta^* \uparrow$

$$\phi(\mathbf{X}) = \begin{cases} 1, & \text{if } T^*(\mathbf{X}) > c^* \Leftrightarrow T(\mathbf{X}) < -c^* \\ \gamma, & \text{if } T^*(\mathbf{X}) = c^* \Leftrightarrow T(\mathbf{X}) = -c^* \\ 0, & \text{if } T^*(\mathbf{X}) < c^* \Leftrightarrow T(\mathbf{X}) > -c^* \end{cases}$$

Q: $-c^* \overset{?}{=} c$

Ans NO

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Example 7.11 (UMP for i.i.d. Bernoulli)

- Let $\mathbf{X} = (X_1, \cdots, X_n)$ be i.i.d. from $B(1, \theta)$, $\theta \in (0, 1)$. Then, the joint pmf is

$$\prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = f(\mathbf{x}|\theta) = \exp\left\{ \log\left(\frac{\theta}{1-\theta}\right) \sum_{i=1}^{n} x_i + n\log(1-\theta) \right\} I_{\{0,1\}^n}(\mathbf{x}).$$

exp(log(·)) $= T(\mathbf{x})$

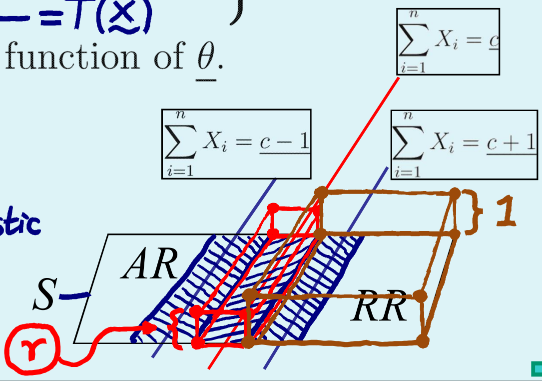
- Here, $C(\theta) = \log\left(\frac{\theta}{1-\theta}\right)$ is a strictly increasing function of θ.
- The level-α UMP test for

$$H_0: \theta \leq \theta_0 \quad \text{vs.} \quad H_A: \theta > \theta_0$$

is given by

reasonable? $\rightarrow$
$$\phi(\mathbf{X}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n X_i > c, \\ \gamma, & \text{if } \sum_{i=1}^n X_i = c, \\ 0, & \text{if } \sum_{i=1}^n X_i < c. \end{cases}$$

test statistic $\sim B(n, \theta)$



- The c and γ are determined by

$$E_{\theta_0}(\phi) = P\left(\sum_{i=1}^n X_i > c \,\middle|\, \theta = \theta_0\right) + \gamma P\left(\sum_{i=1}^n X_i = c \,\middle|\, \theta = \theta_0\right) = \alpha,$$

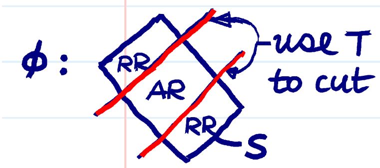
where $\sum_{i=1}^n X_i \sim \text{Binomial}(n, \theta_0)$. ← null distribution.

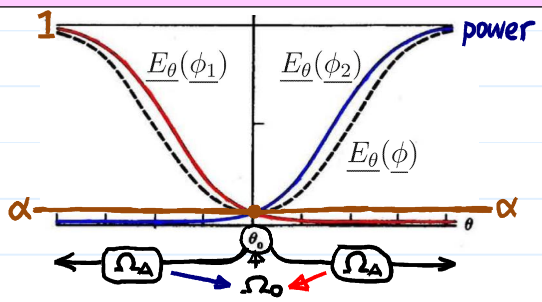
❖ **Reading**: textbook, 9.1, 9.2, 9.2.3; **Further reading**: Roussas, 13.1, 13.2, 13.3

• UMPU tests (for one-parameter exponential family)

Question 7.11

Does a UMP test exist for two-sided alternative hypothesis?

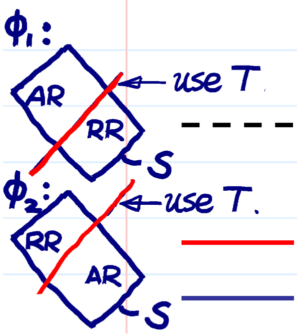




power function $E_\theta(\phi) = P_\theta(RR)$

$\theta \in \Omega_0$ → P(Type I error)

$\theta \in \Omega_A$ → power



ϕ: a "good" test for $H_0: \theta = \theta_0$ vs. $H_A: \theta \neq \theta_0$

Question: How should the power function of ϕ look like?

ϕ_1: UMP for $H_0: \theta = \theta_0$ (or $\theta \geq \theta_0$) vs. $H_A: \theta < \theta_0$

ϕ_2: UMP for $H_0: \theta = \theta_0$ (or $\theta \leq \theta_0$) vs. $H_A: \theta > \theta_0$

They are also level α tests for

Definition 7.8 (unbiased test)

A level-α test function ϕ for $H_0 : \Theta \in \Omega_0$ versus $H_A : \Theta \in \Omega_A$ is said to be **unbiased** if $E_\Theta(\phi)$ satisfies power$_\theta = 1 - \beta_\theta$

α_θ $\quad E_\Theta(\phi) \le \alpha$ if $\Theta \in \Omega_0$ and $E_\Theta(\phi) \ge \alpha$ if $\Theta \in \Omega_A$.

Note. Previous UMP-tests for one-sided hypotheses are unbiased.

Theorem 7.4 (UMPU tests for one-parameter exponential family, two-sided hypothesis)

- Suppose $\mathbf{X} = (X_1, \cdots, X_n)$ have a joint pdf/pmf $f(\mathbf{x}|\theta)$ which has the form of the one-parameter exponential family. ← LNp.23
- Suppose that $C(\theta)$ is a strictly monotone increasing function of θ.
- Null and alternative hypotheses: a special case: $\theta_1 = \theta_2$

$$H_0 : \underbrace{\theta_1 \le \theta \le \theta_2}_{\text{known}} \quad \text{vs.} \quad H_A : \theta < \theta_1 \text{ or } \theta > \theta_2$$

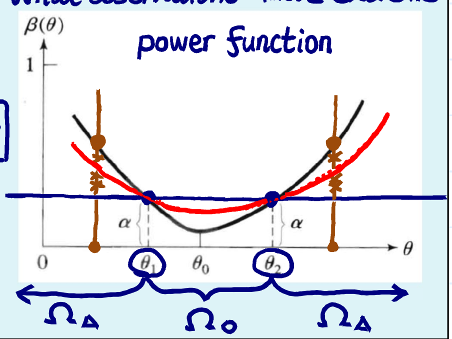
- The level-α uniformly most powerful unbiased test is given by

test statistic

determine what observations "more extreme"

$$\phi(\mathbf{X}) = \begin{cases} 1, & \text{if } T(\mathbf{X}) < c_1 \text{ or } > c_2 \\ \gamma_i, & \text{if } T(\mathbf{X}) = c_i, \quad i = 1, 2 \\ 0, & \text{if } c_1 < T(\mathbf{X}) < c_2 \end{cases} \qquad (1)$$

omitted if $T(x)$ is a continuous r.v.



- The c_1, c_2 and γ_1, γ_2 are determined by

$$E_{\theta_1}[\phi(\mathbf{X})] = E_{\theta_2}[\phi(\mathbf{X})] = \alpha.$$

Use 2 equations to determine $c_1, c_2, \gamma_1, \gamma_2$

Hint. $\frac{\partial}{\partial\theta}\log(f(x|\theta)) = \frac{\frac{\partial}{\partial\theta}f(x|\theta)}{f(x|\theta)}$

$\Rightarrow \frac{\partial}{\partial\theta}f(x|\theta) = \left[\frac{\partial}{\partial\theta}\log(f(x|\theta))\right] f(x|\theta)$

- **Note**. When $\theta_1 = \theta_2$,
  - Null and alternative hypotheses:

simple hypothesis

$$H_0 : \theta = \underbrace{\theta_1}_{\text{known}} \quad \text{against} \quad H_A : \theta \ne \theta_1$$

  - The form of the UMP unbiased test is the same as (1) in LNp.26.
  - The constants c_1, c_2 and γ_1, γ_2 are now determined by

$$E_{\theta_1}[\phi(\mathbf{X})] = \alpha,$$

$\frac{d}{d\theta}E_\theta(\phi)\Big|_{\theta=\theta_1} = 0$ (check the plot in LNp.26)

and

$$E_{\theta_1}[\phi(\mathbf{X})T(\mathbf{X})] = E_{\theta_1}[\phi(\mathbf{X})]\, E_{\theta_1}[T(\mathbf{X})] = \alpha\, E_{\theta_1}[T(\mathbf{X})].$$

$\phi(x)$ & $T(x)$ are uncorrelated when $\theta = \theta_1$

Example 7.12 (UMPU test for normal mean)

- Let $\mathbf{X} = (X_1, \cdots, X_n)$ be i.i.d. from $N(\mu, \sigma^2)$, where σ is known. → μ: parameter
- The joint pdf is

$$f(\mathbf{x}|\mu) = \exp\left\{\underbrace{\frac{n\mu}{\sigma^2}}_{C(\mu)}\underbrace{\bar{x}}_{T(x)} - \frac{2n\mu^2}{\sigma^2} - \frac{\sum_{i=1}^n x_i^2}{2\sigma^2} - n\log(\sqrt{2\pi})\right\}.$$

which belongs to one-parameter exponential family. → $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

- In this case, $C(\mu) = n\mu/\sigma^2$, which is an increasing function of μ.
- Null and alternative hypotheses:

$$H_0 : \mu = \underbrace{\mu_0}_{\text{known}} \quad \text{vs.} \quad H_A : \mu \ne \mu_0$$

where μ_0 is a given value.

- The level-α UMP unbiased test is given by — a continuous r.v. under H_0

test statistic

$$\phi(\mathbf{X}) = \begin{cases} 1, & \text{if } \overline{X} < c_1 \text{ or } > c_2, \\ 0, & \text{if } c_1 < \overline{X} < c_2. \end{cases}$$

Q: Why not use a randomized test?

- The c_1, c_2 are determined by — $N(\mu_0, \frac{\sigma^2}{n})$ under H_0

$$E_{\mu_0}[\phi(\mathbf{X})] = \alpha, \quad E_{\mu_0}[\phi(\mathbf{X})\,\overline{X}] = \alpha\, E_{\mu_0}(\overline{X}) = \mu_0 \cdot E_{\mu_0}(\phi(\underset{\sim}{X})) = E_{\mu_0}[\mu_0 \cdot \phi(\underset{\sim}{X})]$$

$$\Rightarrow E_{\mu_0}[\phi(\underset{\sim}{X})\cdot(\overline{X}-\mu_0)] = 0$$

- Further derivation:
  - The $\phi(\mathbf{X})$ can be expressed as

$$\Rightarrow E_{\mu_0}\Big[\phi(\underset{\sim}{X})\cdot \underbrace{\frac{\sqrt{n}(\overline{X}-\mu_0)}{\sigma}}_{\equiv Z}\Big] = 0$$

test statistic

$$\phi(\mathbf{X}) = \begin{cases} 1, & \text{if } \underbrace{\frac{\sqrt{n}(\overline{X}-\mu_0)}{\sigma}}_{=Z} < c_1' \text{ or } > c_2', \\ 0, & \text{if } c_1' < \frac{\sqrt{n}(\overline{X}-\mu_0)}{\sigma} < c_2'. \end{cases}$$

where $c_i' = \sqrt{n}(c_i - \mu_0)/\sigma$. $= Z \sim N(0,1)$ under H_0,

$$0 = \int_{-\infty}^{c_1'} + \int_{c_2'}^{\infty} z \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{-\infty}^{c_1'} - \int_{-\infty}^{-c_2'} z \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

  - Because the pdf of $N(0,1)$ is symmetry about 0, $c_1' = -c_2'$. $\cancel{c_1 = c_2}$
  - Therefore, we have

reasonable? →

$$\phi(\mathbf{X}) = \begin{cases} 1, & \text{if } \left[(\overline{X}-\mu_0)/(\sigma/\sqrt{n})\right]^2 > c, \\ 0, & \text{otherwise.} \end{cases}$$

why need this? — ① Note. $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$ ② remove unit

  - The constant c is determined by $P(\chi_1^2 > c) = \alpha$ since under H_0,

$$\left[(\overline{X}-\mu_0)/(\sigma/\sqrt{n})\right]^2 \sim \chi_1^2. \text{ — null distribution}$$