• Neyman Pearson Lemma --- most powerful tests

Recall (LNp 7) power $= P(RR|\theta)$, $\theta \in \Omega_A$ $= 1 - P(AR|\theta) = 1 - \beta_\theta$

Question 7.8

Among all tests with significance level α, what is the "best" test? criterion=?

level-α tests

S — AR | RR

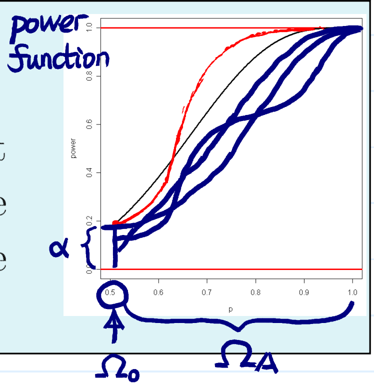
Recall. Two types of errors — α_θ, $\theta \in \Omega_0$ ← protect; β_θ, $\theta \in \Omega_A$

Definition 7.6 (uniformly most powerful test, TBp. 336) may not exist in some cases

For testing

$$H_0 : \Theta \in \Omega_0 \quad \text{vs.} \quad H_A : \Theta \in \Omega_A,$$

a test is called **uniformly most powerful (UMP)** test with significance level α, if $\forall\, \Theta \in \Omega_A$ the power of the test is larger than (i.e., β_Θ is smaller) or equal to the power of any other tests with significance level α.



• most powerful test for simple vs. simple hypotheses

Definition 7.7 (likelihood ratio, TBp.329)

- let $\mathbf{X} = (X_1, \ldots, X_n)$ have a joint pmf/pdf f, where $f \in \Omega = \{f_0, f_A\}$.
- null and alternative hypotheses: $H_0 : f_0$ vs. $H_A : f_A$

- likelihood ratio: $\dfrac{f_0(\mathbf{x})}{f_A(\mathbf{x})}$

a way to numerically evaluate the degree of "how extreme"

smaller value ⇒ "more extreme" ⇒ "more support on H_A"

Q: Why not use $f_0(x) - f_A(x)$?

MLE. $f_0(x) \gtrless f_A(x)$; $f_0(x)/f_A(x) \gtrless 1$; $f_0(x) - f_A(x) \gtrless 0$

Question: We should reject H_0 when $f_0(\mathbf{x})/f_A(\mathbf{x})$ is large or small? Why?

Theorem 7.1 (Neyman-Pearson lemma, TBp. 332)

Suppose that the likelihood ratio test that rejects H_0 when

$$0 \le f_0(\mathbf{x})/f_A(\mathbf{x}) < c$$

evaluate what observations are "more extreme"

S — AR | RR; other test stat.; use $\frac{f_0}{f_A} < c$ to cut

has significance level α. Then any other test which has significance level $\alpha^* \le \alpha$ has power less than or equal to the power of the likelihood ratio test.

Proof. (*for continuous case*) Let $R^\dagger$ be the rejection region of the likelihood ratio test, and R be the rejection region of any other test with significance level $\alpha^* \le \alpha$. Then,

$$\int_{R^\dagger} f_0(x)\,dx = \alpha \ge \alpha^* = \int_R f_0(x)\,dx,$$

null distribution

LR test ⇔ MP test

and hence the difference of the powers is

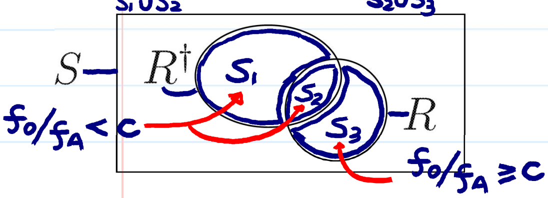
$$\underbrace{\int_{R^\dagger}}_{S_1 \cup S_2} f_A(x)\,dx - \underbrace{\int_R}_{S_2 \cup S_3} f_A(x)\,dx = \int_{R^\dagger \setminus R} f_A(x)\,dx - \int_{R \setminus R^\dagger} f_A(x)\,dx \quad -\int_{S_2} \tfrac{1}{c} f_0(x)dx$$

$$\ge \int_{R^\dagger \setminus R} \tfrac{1}{c} f_0(x)\,dx - \int_{R \setminus R^\dagger} \tfrac{1}{c} f_0(x)\,dx + \int_{S_2} \tfrac{1}{c} f_0(x)dx$$

$$= \tfrac{1}{c}\int_{R^\dagger} f_0(x)\,dx - \tfrac{1}{c}\int_R f_0(x)\,dx \ge 0.$$

S_1, S_3

$\because f_A(x) \ge \frac{1}{c} f_0(x)$, for $x \in S_1$; $f_A(x) \le \frac{1}{c} f_0(x)$, for $x \in S_3$



Example 7.7 (cont. Ex. 7.2 (LNp.8), TBp. 329, 330)

If $X \sim \text{Binomial}(10, p)$ and consider testing

$$H_0: p = 0.5 \quad \text{vs.} \quad H_A: p = 0.7.$$

Note. $f_0(x)/f_A(x) \downarrow \Leftrightarrow x \uparrow$
$\Rightarrow f_0(x)/f_A(x) < C \Leftrightarrow x > x_0$

The likelihood ratios are — more support on H_A → RR

x	0	1	2	3	4	5	6	7	8	9	10
$B(10,0.5) \rightarrow f_0(x)$	.001	.010	.044	.117	.205	.246	.205	.117	.044	.010	.001
$B(10,0.7) \rightarrow f_A(x)$	.0000	.0001	.001	.009	.037	.103	.200	.267	.234	.121	.028
$f_0(x)/f_A(x)$	165.4	70.9	30.4	13.0	5.58	2.39	1.03	.44	.19	.08	.03

smaller

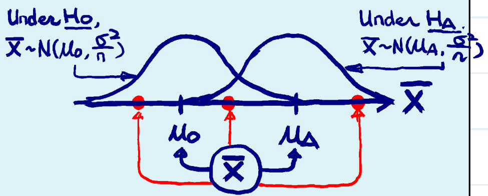
and the likelihood ratio test rejects H_0 for small values of $f_0(x)/f_A(x)$ which corresponding to large values of X. cf. Q: What if we use $f_0(x) - f_A(x)$?

Example 7.8 (TBp. 333)

- Let $X_1, X_2, \ldots, X_n$ be i.i.d. from $N(\mu, \sigma^2)$ with σ^2 known. → parameter : μ.

$$H_0: \mu = \mu_0 \quad \text{vs.} \quad H_A: \mu = \mu_A, \rightarrow \Omega = \{N(\mu_0, \sigma^2), N(\mu_A, \sigma^2)\}$$

where μ_0, μ_A are given constants. ← known



- Suppose the significance level is prescribed as α.
- The likelihood ratio is

$$\frac{f_0(\mathbf{x})}{f_A(\mathbf{x})} = \frac{\cancel{\otimes}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^n (X_i - \mu_0)^2\right]}{\cancel{\otimes}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^n (X_i - \mu_A)^2\right]} \propto \exp\left[2n\,\bar{X}\,(\mu_0 - \mu_A) + n\mu_A^2 - n\mu_0^2\right].$$

transformation of data.

check ★ in LNp.11

- Suppose $\mu_0 - \mu_A < 0$ ($\mu_A > \mu_0$). Likelihood ratio is small $\Leftrightarrow$ $\bar{X}$ is large.
- Thus, the most powerful test rejects H_0 for — more extreme ⇒ larger $\bar{X}$

reason-able?

test statistic → $\bar{X} > x_0$ for some x_0. — the form of RR

- The x_0 is chosen so that the test has the desired level α

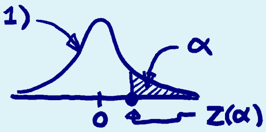
critical value

$\bar{X} \sim N(\mu_0, \sigma^2/n)$

$$\alpha = P(\bar{X} > x_0 \,|\, H_0) = P(\bar{X} > x_0 \,|\, \mu = \mu_0) = P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > \frac{x_0 - \mu_0}{\sigma/\sqrt{n}}\right).$$

Note. 1. x_0 irrelevant to μ_A
2. x_0 could be larger than μ_A

What's the null distribution? $\sim N(0.1)$



- Hence, solve

$$x_0 = \mu_0 + z(\alpha)\cdot\frac{\sigma}{\sqrt{n}} \Leftarrow \frac{x_0 - \mu_0}{\sigma/\sqrt{n}} = z(\alpha)$$

$\alpha \downarrow$, $z(\alpha) \uparrow$
$x_0 \uparrow$, RR $\downarrow$

for x_0, where $z(\alpha)$ is the $1 - \alpha$ quantile of the $N(0,1)$ distribution.

- Question: What if $\mu_0 > \mu_A$ (i.e., $\mu_A < \mu_0$)?

Question 7.9

Why not just reject H_0 when $f_0(\mathbf{x})/f_A(\mathbf{x}) \le 1$? $\Leftrightarrow f_A(x) > f_0(x)$ ← cf. MLE

Note. Asymmetry between H_0 & H_A

e.g. if $c = 1/100$,
RR: $f_A(x) > 100 \cdot f_0(x)$ — ∵ α small

N-P Lemma
$0 < \frac{f_0(x)}{f_A(x)} < C \Leftrightarrow f_A(x) > \frac{1}{C} f_0(x)$
C small $\Leftrightarrow$ α small

- UMP examples for testing certain composite hypotheses

Example 7.9 (cont. Ex. 7.8 (LNp.18), TBp.336)

- Let $X_1, X_2, \ldots, X_n$ be i.i.d. from $N(\mu, \sigma^2)$, σ^2 known.

$\Omega = \{\mu \mid \mu \geq \mu_0\}$ simple ← $H_0^{(2)}: \mu = \mu_0$ vs. $H_A^{(2)}: \mu > \mu_0$ → composite, one-sided

- In Ex. 7.8, for any particular simple alternative H_A: $\mu = \mu_A$ $(> \mu_0)$, the UMP test rejects H_0 for

S – AR / RR ← use $\overline{X} > x_0$ ← $\mu_0 + z(\alpha)\frac{\sigma}{\sqrt{n}}$

$$\overline{X} > x_0,$$

where x_0 **does not depend on the value of** μ_A.

$H_0: \mu = \mu_0$ vs. $H_A: \mu = \mu_A$
same most powerful test for any $\mu_A > \mu_0$

- Thus, this test is also UMP for $H_A^{(2)}: \mu > \mu_0$.

the order of extremity is identical for any μ_A

Example 7.10 (cont. Ex. 7.9, TBp.336)

- Let $X_1, X_2, \ldots, X_n$ be i.i.d. from $N(\mu, \sigma^2)$, σ^2 known. Consider testing

$$H_0^{(3)}: \mu \leq \mu_0 \quad \text{vs.} \quad H_A^{(3)}: \mu > \mu_0. \quad \longrightarrow \Omega = \{\mu \mid \mu \in \mathbb{R}\}$$

- **Claim**: The test $\phi^*(\mathbf{x})$ which rejects H_0 when

LNp.6 UMP in Ex.7.9

$$\overline{X} > x_0,$$

where x_0 is determined by:

for any $\mu \leq \mu_0$

$$\alpha = P(\overline{X} > x_0 \mid \mu = \mu_0) = P\left(\underbrace{\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}}_{\sim N(0,1) \text{ when } \mu = \mu_0} \geq \frac{x_0 - \mu_0}{\sigma/\sqrt{n}} \,\middle|\, \mu = \mu_0\right)$$

$x_0 = \mu_0 + z(\alpha)\frac{\sigma}{\sqrt{n}}$

is an level-α UMP test for $H_0^{(3)}$ vs. $H_A^{(3)}$.

- This follows from
  1. ϕ^* is a level-α test because for $\mu \leq \mu_0$, ($\mu \in \Omega_0$)

cf. level-α test in Ex 7.8 (LNp.18~19)

$$P(RR \mid \mu) = P(\overline{X} > x_0 \mid \mu) = P\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \geq \frac{x_0 - \mu}{\sigma/\sqrt{n}} \,\middle|\, \mu\right)$$

$\sim N(0,1)$ when the mean is μ

$P(RR \mid \mu) = \alpha_\mu$; $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$

$\alpha_\mu \downarrow$ when $\mu \downarrow$

$$= 1 - \Phi\left(\frac{x_0 - \mu}{\sigma/\sqrt{n}}\right) \leq 1 - \Phi\left(\frac{x_0 - \mu_0}{\sigma/\sqrt{n}}\right) = \alpha.$$

cdf of $N(0,1)$ $\quad = z(\alpha)$

  2. ϕ^* is UMP because if there is another level-α test $\phi(\mathbf{x})$ for testing

$$H_0^{(3)}: \mu \leq \mu_0 \text{ v.s. } H_A^{(3)}: \mu > \mu_0,$$

then ϕ is also a level-α test for

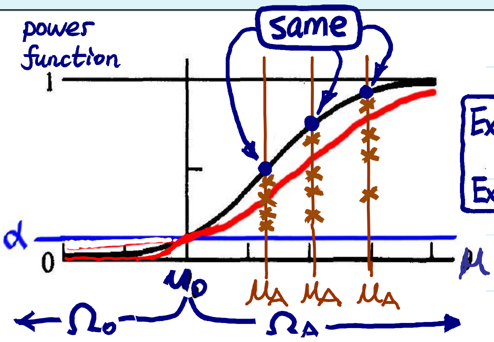
Ex.7.9 (LNp.20) → $$H_0^{(2)}: \mu = \mu_0 \text{ v.s. } H_A^{(2)}: \mu > \mu_0,$$

and ϕ is always less powerful than ϕ^*.

Note. All composite hypotheses discussed here are one-sided

By N-P Lemma, decide a UMP test ϕ_{μ_A} for

Plot of $P(RR|\theta) = E_\theta(\phi)$ against θ



Ex.7.8 ~ Ex.7.10

$H_0^{(1)}: \mu = \mu_0$ vs. $H_A^{(1)}: \mu = \mu_A$

← show ϕ_{μ_A} irrelevant to μ_A, for $\mu_A \in \Omega_A$, $\phi_{\mu_A} \to \phi$

$H_0^{(2)}: \mu = \mu_0$ vs. $H_A^{(2)}: \mu > \mu_0$

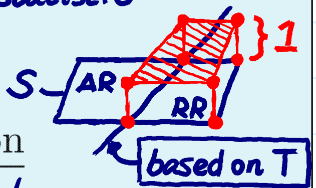
← show $P_\mu(RR_\phi) \leq \alpha$, for $\mu \in \Omega_0$

$H_0^{(3)}: \mu \leq \mu_0$ vs. $H_A^{(3)}: \mu > \mu_0$

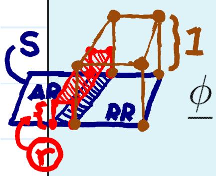
Definition 7.7 (test function, nonrandomized test, randomized test)

- For a (nonrandomized) test, its **test function** $\phi(\mathbf{x})$ is *of → test statistic*; *a statistic*
  - a function defined on sample space S and
  - taking values in $[0, 1]$ such that

indicator function $\;\; I_{RR}(\mathbf{x}) = \phi(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \text{rejection region} \\ 0, & \text{if } \mathbf{x} \in \text{acceptance region} \end{cases}$



- A *randomized* test function $\phi(\mathbf{x})$: ← *mainly for discrete case (see Ex.7.4, LNp.15)*
  - It has a form of:



$$\phi(\mathbf{x}) = \begin{cases} \boxed{1}, & \text{if } \mathbf{x} \in \text{rejection region} \\ \boxed{\gamma}, & \text{if } \mathbf{x} \in \text{boundry of rejection and acceptance regions} \\ \boxed{0}, & \text{if } \mathbf{x} \in \text{acceptance region} \end{cases}$$

where $0 < \gamma < 1$. ← *rejection probability*
  - When $\phi(\mathbf{x}) = \gamma$, a coin, whose probability of falling head is γ, is tossed and H_0 is rejected or accepted when head or tail appears, respectively. ← *not required if test statistic is a continuous r.v. under H_0*

Theorem 7.2 (significance level, power, and test function)

$$P_\theta(RR) = E_\Theta[\phi(\mathbf{X})] = \begin{cases} \alpha_\Theta, & \text{for } \Theta \in \Omega_0 \;\to P_\theta(\text{Type I error}) \\ 1 - \beta_\Theta, & \text{for } \Theta \in \Omega_A \;\to \text{power}_\theta \end{cases}$$

- UMP tests for one-parameter exponential family

Recall. A one-parameter exponential family has the joint pmf/pdf of the form

dim = 1 $\;\; f(\mathbf{x}|\theta) = \exp\{C(\theta)\,T(\mathbf{x}) + d(\theta) + S(\mathbf{x})\} I_A(\mathbf{x}), \;\to \dfrac{f(\mathbf{x}|\theta_0)}{f(\mathbf{x}|\theta_A)} \propto \exp\{T(\mathbf{x})[C(\theta_0)-C(\theta_A)]\}$

where the set A is independent of θ. *Recall. Exponential family is useful in identifying suff. + complete statistics*

Theorem 7.3 (UMP test for one-parameter exponential family, one-sided hypothesis)

- Suppose $\mathbf{X} = (X_1, \cdots, X_n)$ have a joint pdf/pmf $f(\mathbf{x}|\theta)$ which has the form of the one-parameter exponential family.
- Suppose that $C(\theta)$ is a strictly monotone increasing function of θ.

Q What if $C(\theta)$ is decreasing? Note. $C(\theta)T(\mathbf{x}) = [-C(\theta)][-T(\mathbf{x})]$, $= C^*(\theta)$, $= T^*(\mathbf{x})$

- Then the level-α UMP test for testing

$$H_0: \theta \leq \theta_0 \quad (\text{or } H_0: \theta = \theta_0) \quad \text{vs.} \quad H_A: \theta > \theta_0$$

$C(\theta) \leq C(\theta_0)$ $\;\;$ *$C(\theta) > C(\theta_0)$*

Why need it? In LNp.18, replace $\bar{X}$ by $T(\mathbf{X})$, replace $\mu_0 - \mu_A$ by $C(\theta_0) - C(\theta_A)$

is given by

$H_A: \theta > \theta_0$, larger T more extreme test statistic

$$\phi(\mathbf{X}) = \begin{cases} 1, & \text{if } \boxed{T(\mathbf{X})} > c, \\ \gamma, & \text{if } \boxed{T(\mathbf{X})} = c, \\ 0, & \text{if } \boxed{T(\mathbf{X})} < c. \end{cases}$$

- The c and γ are determined by *For continuous r.v., $P(T(\mathbf{X}) = c) = 0$*

(exercise) $\;\; E_{\theta_0}(\phi) = P(T(\mathbf{X}) > c \,|\, \theta = \theta_0) + \gamma\, P(T(\mathbf{X}) = c \,|\, \theta = \theta_0) = \alpha.$

Proof. The proof follows the same argument as in Examples 7.8 to 7.10.

Question 7.10

What if we are interested in $H_0 : \theta \geq \theta_0$ vs. $H_A : \theta < \theta_0$?

$H_A: \theta < \theta_0$
smaller T
more extreme

$$\theta^* = -\theta$$
$$C^*(\theta^*) = -C(-\theta^*)$$
$$T^*(\mathbf{x}) = -T(\mathbf{x})$$

$C^* T^* = C T$

C^*: an increasing function of θ^*

$H_0 : \theta^* \leq -\theta_0$ vs. $H_A : \theta^* > -\theta_0$ | $C^*(\theta^*) \uparrow$, as $\theta^* \uparrow$

$$\phi(\mathbf{X}) = \begin{cases} 1, & \text{if } T^*(\mathbf{X}) > c^* \Leftrightarrow T(\mathbf{X}) < -c^* \\ \gamma, & \text{if } T^*(\mathbf{X}) = c^* \Leftrightarrow T(\mathbf{X}) = -c^* \\ 0, & \text{if } T^*(\mathbf{X}) < c^* \Leftrightarrow T(\mathbf{X}) > -c^* \end{cases}$$

Q: $-c^* \overset{?}{=} c$
Ans NO

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Example 7.11 (UMP for i.i.d. Bernoulli)

- Let $\mathbf{X} = (X_1, \cdots, X_n)$ be i.i.d. from $B(1, \theta)$, $\theta \in (0,1)$. Then, the joint pmf is

$$\prod_{i=1}^n \theta^{x_i}(1-\theta)^{1-x_i} = f(\mathbf{x}|\theta) = \exp\left\{ \underbrace{\log\left(\frac{\theta}{1-\theta}\right)} \underbrace{\sum_{i=1}^n x_i}_{=T(\mathbf{x})} + n\log(1-\theta) \right\} I_{\{0,1\}^n}(\mathbf{x}).$$

$\exp(\log(\cdot))$

- Here, $C(\theta) = \log\left(\frac{\theta}{1-\theta}\right)$ is a strictly increasing function of θ.
- The level-α UMP test for

$$H_0 : \theta \leq \theta_0 \quad \text{vs.} \quad H_A : \theta > \theta_0$$

is given by

$$\phi(\mathbf{X}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n X_i > c, \\ \gamma, & \text{if } \sum_{i=1}^n X_i = c, \\ 0, & \text{if } \sum_{i=1}^n X_i < c. \end{cases}$$

reason-able ?

test statistic