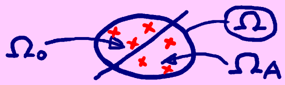
Hypothesis testing ← 假設檢定

• What is hypothesis testing?

Question 7.1 (What is a hypothesis testing question?)

1. **observed data.** $x_1, \ldots, x_n$
2. **statistical modeling.** Regard $x_1, \ldots, x_n$ as a realization of random variables $X_1, \ldots, X_n$, and assign $X_1, \ldots, X_n$ a joint distribution: ← statistical modeling
   a joint cdf $F(\cdot|\Theta)$, or a joint pdf $f(\cdot|\Theta)$, or a joint pmf $p(\cdot|\Theta)$,
   parameters 規律 where $\Theta = (\theta_1, \ldots, \theta_k) \in \Omega$, and θ_i's are **fixed constants**, but their values are **unknown**. parameter space | Note. Ω can be regarded as a collection of distributions. or 平行世界
3. **point estimation.** What is the value of Θ? Find a function of $X_1, \ldots, X_n$, $\widehat{\Theta}$, to estimate Θ or a function of Θ. point estimator
4. **hypothesis testing.** Separate Ω into two disjoint sets Ω_0 and Ω_A, i.e.,



$$\Omega_0 \cap \Omega_A = \emptyset \quad \text{and} \quad \Omega_0 \cup \Omega_A = \Omega.$$ ← a partition of Ω.

   Use the data $X_1, \ldots, X_n$ to answer the question: which of the two hypotheses,

$$H_0 : \Theta \in \Omega_0 \quad \text{versus} \quad H_A : \Theta \in \Omega_A$$

   is more favorable. true parameter

Question 7.2

Can we obtain an estimate of Θ and accept H_0 if $\hat{\Theta} \in \Omega_0$ and reject H_0 if $\hat{\Theta} \in \Omega_A$? What if X is continuous?

Hint. Consider the case: $X \sim \text{Binomial}(n, p)$, $\boxed{0<p<1}$

fair coin $\quad H_0 : p = 0.5 \quad \text{v.s.} \quad H_1 : p \neq 0.5,$

What if $n = 10^5$ and $\hat{p} = 0.50001$? ← $\boxed{\neq 0.5}$

testing $\xleftrightarrow{\text{duality}}$ C.I. (interval estimator) LNp.54~57

Definition 7.1 (null and alternative hypotheses, simple and composite hypotheses, TBp.331,332,334)

- H_0 is called **null hypothesis**. ← 虛無假設
- H_A (sometimes denoted as H_1) is called **alternative hypothesis**. ← 對立假設
- An hypothesis is said to be a **simple hypothesis** if that hypothesis uniquely specifies the distribution. ← Ω_0 or Ω_A only contain 1 distribution
- Any hypothesis that is not a simple hypothesis is called a **composite hypothesis**. ← Ω_0 or Ω_A contain at least 2 distributions.

Question 7.3 (asymmetry between H_0 and H_A)

Is there a difference between the roles of H_0 and H_A? Can we arbitrary exchange the two hypotheses? → No

→ Ans. Yes. See the solution to dilemma (LNp.8)

Example 7.1 (some null and alternative hypotheses, TBp.329, 334)

1. Two coins experiment
   - Data and problem
     - Suppose that I have two coins
     - Coin 0 has probability of heads equal to 0.5, and coin 1 has probability of heads equal to 0.7. ← parameter
     - I choose one of the coins, toss it 10 times, and tell you the number of heads X ← data
     - On the basis of observed X, your task is to decide which coin it was. ← problem
   - Statistical modeling. unknown parameter for you
     - $X \sim \text{Binomial}(10, p)$
     - $\Omega = \{\text{Binomial}(10, 0.5), \text{Binomial}(10, 0.7)\} = \{p \mid p \in \{0.5, 0.7\}\}$

     Note. not $0 < p < 1$ in the case
   - Problem formulation — Ω can be regarded as the collection of "all possible worlds." In this case, there are only two possible worlds.
     - H_0: coin 0 $\Rightarrow \Omega_0 = \{\text{Binomial}(10, 0.5)\}$ ← dim=0
     - H_A: coin 1 $\Rightarrow \Omega_A = \{\text{Binomial}(10, 0.7)\}$ ← dim=0
     - Both hypotheses are simple.

2. Testing for ESP ← Extra-Sensory Perception
   - Data and problem
     - A subject is asked to identify, without looking, the suits of 20 cards drawn randomly with replacement from a 52 card deck.
     - Let T be the number of correct identifications. ← Data
     - We would like to know whether the person is purely guessing or has extrasensory ability. ← problem
   - statistical modeling — unknown parameter
     - $T \sim \text{Binomial}(20, p)$.
     - Note that $p = 0.25$ means that the subject is mearly guessing and has no extrasensory ability.
     - $\Omega = \{p : p \in [0.25, 1]\}$ or $\Omega = \{p : p \in [0, 1]\}$

guess \ card	S (1/4)	H (1/4)	D (1/4)	C (1/4)
g_1 S	V	X	X	X
g_2 H	X	V	X	X
g_3 D	X	X	V	X
g_4 C	X	X	X	V

purely guessing: independent

prob. $= \frac{1}{4} g_i$

   What's their difference? → When T is very small.

   - Problem formulation
     - $\Omega = [0.25, 1]$

       $H_0 : p = 0.25$ v.s. $H_A : p > 0.25$

       dim=0 dim=1
       * Then, $\Omega_0 = \{0.25\}$ and $\Omega_A = (0.25, 1]$.
       * The H_0 is simple and H_A is composite. Furhermore, H_A is called a **one-sided hypothesis**.

– $\Omega = [0,1]$

$H_0 : p = 0.25$ v.s. $H_A : p \neq 0.25$

dim=0 dim=1

∗ Then, $\Omega_0 = \{0.25\}$ and $\Omega_A = [0, 0.25) \cup (0.25, 1]$.

∗ The H_0 is simple and H_A is composite. Furhermore, H_A is called a **two-sided hypothesis**.

3. *goodness-of-fit test* (for Poisson distribution)

Taylor expansion of mgf: $M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} \frac{E(X^k)}{k!} t^k$

kth moment of X: parameter

- Data and problem (LN, CH8, p.3-68, χ^2-test)
  - Observe an i.i.d. data $X_1, \ldots, X_n$. Data
  - Suppose that we only know that X_i's are discrete data. $\in \{0, 1, 2, 3, \cdots\}$
  - We would like to know whether the observations came from a Poisson distribution. problem

dim=?

$0, 1, 2, 3, \cdots$

$P_0\ P_1\ P_2\ P_3 \cdots$ s.t. $\sum_{i=0}^{\infty} P_i = 1$

dim=∞

- Statistical modeling:
  - $X_1, \ldots, X_n$ are i.i.d. from a discrete distribution with cdf F. on non-negative integers
  - $\Omega = \{F : F \text{ is a discrete cdf}\}$ unknown distribution
- Problem formulation on $\{0, 1, 2, 3, \cdots\}$
  - H_0 : Data came from some Poisson $\Rightarrow \Omega_0 = \{F : F \text{ is a } P(\lambda) \text{ cdf}\}$ dim=1
  - H_A : Data not from Poisson $\Rightarrow \Omega_A = \Omega \setminus \Omega_0$.
  - Both hypotheses are composite. dim=∞