Interval Estimation ← cf. → Point Estimation

• What is interval estimation?

Question 6.10

- Is it satisfactory to report only an estimated value of θ? → point estimation
- Note that
  1. A point estimate, although it will represent our best guess for the true value of the parameter, may be close to that true value but will virtually *never equal* it. e.g. $P(\hat{\theta}=\theta)=0$, if $\hat{\theta}$ is a continuous r.v.
  2. Some measure of how close the point estimate is to the true value is required. One way to do this is to report both the *estimate* and its *estimated standard error*.
- The following questions arise naturally:
  1. A point estimator only gives a value. Wound it be better that we can give customer a range of possible values? This amounts to replacing the point estimate, a single value, by an entire interval of plausible values. → interval estimation.
  2. Is there an estimation method that can combine together the two types of information, i.e., estimated value and estimated standard error?

Definition 6.22 (interval estimator, coverage probability, interval estimate, confidence interval, and confidence level, TBp. 217 & 279)

- For a random vector $\mathbf{X}=(X_1,\ldots,X_n)$, an **interval estimator** of a parameter θ with **coverage probability** $1-\alpha$ is a *random interval*

  data, treated as r.v.'s — assign $\mathbf{X}$ a joint dist. $0<\alpha<1$

$$(\hat{\theta}_L(\mathbf{X}), \hat{\theta}_U(\mathbf{X})),$$

  statistics (random variables)

  where
  1. $\hat{\theta}_L(\mathbf{X})$, $\hat{\theta}_U(\mathbf{X})$ are functions of data,
  2. $\hat{\theta}_L(\mathbf{X}) < \hat{\theta}_U(\mathbf{X})$, and,
  3. $P(\theta \in (\hat{\theta}_L(\mathbf{X}), \hat{\theta}_U(\mathbf{X}))) = 1-\alpha$.

  cf. $P(X\in(a,b))$ — random, fixed

  Sometimes, defined as "$\geq$"

  probability that the interval estimator will cover θ

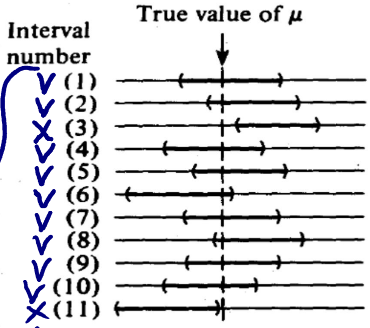
- If $\mathbf{X}=\mathbf{x}$ is observed, the interval $(\hat{\theta}_L(\mathbf{x}), \hat{\theta}_U(\mathbf{x}))$ is called an **interval estimate**. values — values
- The term "$100\cdot(1-\alpha)\%$ **confidence interval**" is used to denote either an interval estimator with coverage probability $1-\alpha$ or an interval estimate. 信賴區間 — same
- The $100(1-\alpha)\%$ is also referred to as **confidence level**. 信賴水準
- **Note.** The α is usually assigned a small value, e.g. 0.1, 0.05, or 0.01.

  90% — 95% — 99%

95% correct on average

about 950 V, 50 X

interval estimator



(1000)

Repeated construction of 95% confidence intervals

Question 6.11

How to interpret the $100(1-\alpha)\%$ in a $100(1-\alpha)\%$ interval *estimate* $(\hat{\theta}_L(\mathbf{x}), \hat{\theta}_U(\mathbf{x}))$? For example, for a 95% interval estimate, say $(23.5, 47.8)$, can we say that the probability is either 0 or 1.

θ not random $\quad P(\theta \in (23.5, 47.8)) = 0.95$? NO.

• methods for constructing interval estimators

Recall. methods for the estimation of standard error

LNp.12

When (i) sample size is large enough, (ii) difficult to get (1)

When (i) sample size is not large enough, and (ii) difficult to get (1)

(1) exact distribution (2) asymptotic method (3) bootstrap → sampling distribution

The three methods can be applied to construct confidence intervals.

Definition 6.23 (pivotal quantity)

e.g., $N(\mu, \sigma^2)$, $\theta = \mu$, all parameters: (μ, σ^2)

A function of data $X_1, \ldots, X_n$ and parameter θ, denoted by

Note We may have more parameters in addition to θ.

cf. r.v. $Q(\mathbf{X}, \theta) = Q(X_1, \ldots, X_n, \theta)$, to construct C.I. for θ

is called a **pivotal quantity** for θ if the distribution of $Q(\mathbf{X}, \theta)$ is irrelevant to *all* parameters. (cf. ancillary statistics) ← LNp.52 cf. not a statistic

Example 6.38 (some pivotal quantities, TBp. 279-281)

Poisson(λ) cf. → mean $1/\lambda$

1. Let $X_1, \ldots, X_n$ be i.i.d. from Exponential distribution $E(\lambda)$, then unknown parameter

(Ex) MLE of λ: $1/\overline{X}$

$\sum_{i=1}^n X_i \sim Gamma(n, \lambda)$

$$\lambda \sum_{i=1}^n X_i \sim \Gamma(n, 1).$$

LN, CH1~6, p.74, items 4 & 6.

Because the pdf of $\Gamma(n, 1)$ is irrelevant to λ, $\lambda \sum_{i=1}^n X_i$ is a pivotal quantity. a function of data & λ

2. Let $X_1, \ldots, X_n$ be i.i.d. from Uniform distribution $U(0, \theta)$. Then, the pdf of unknown parameter

$Y_i = \frac{X_i}{\theta} \overset{\text{i.i.d.}}{\sim} U(0,1)$, $\frac{X_{(n)}}{\theta} = Y_{(n)} = \max\{Y_1, \cdots, Y_n\}$

$$Q = \frac{X_{(n)}}{\theta}$$

MLE of θ: $X_{(n)}$

is

$$nq^{n-1}, \quad 0 \le q \le 1.$$

a function of data & θ

Because the pdf is irrelevant to θ, $X_{(n)}/\theta$ is a pivotal quantity.

3. Let $X_1, \ldots, X_n$ be i.i.d. from Normal distribution $N(\mu, \sigma^2)$, where μ is a parameter and the value of σ is known. Then, unknown parameter

σ: not a parameter

MLE of μ: $\overline{X}$

$\overline{X} \sim N(\mu, \sigma^2/n)$ standardization

$$\frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \sim N(0, 1).$$

LN, CH1~6, P.80, item 1

Because the pdf of $N(0, 1)$ is irrelevant to μ, $\frac{\sqrt{n}(\overline{X}-\mu)}{\sigma}$ is a pivotal quantity.

not a pivotal quantity for μ in case 4 (LNp.79) a function of data & μ