

Example 6.33 (cont. Ex. 6.32, UMVUE of Poisson mean)

Suppose X_1, \dots, X_n is an i.i.d. sample from $\text{Poisson}(\lambda)$ distribution. Then $S = \sum_{i=1}^n X_i$ is a sufficient and complete statistics for λ . Because $\bar{X} = S/n$ is a function of S and $E(\bar{X}) = \lambda$, \bar{X} is UMVUE of λ . $S \sim P(n\lambda) \Rightarrow E(S) = n\lambda$

Example 6.34 (UMVUE of Normal mean and variance)

Let X_1, \dots, X_n be i.i.d. random variables from Normal distribution $N(\mu, \sigma^2)$, then \bar{X} and $S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$ are sufficient and complete statistics for (μ, σ^2) . Clearly, $E(\bar{X}) = \mu$ and $E(S^2) = \sigma^2$ (Note. $(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2 \Rightarrow E((n-1)S^2/\sigma^2) = n-1$), so \bar{X} and S^2 are unbiased estimator of μ and σ^2 , respectively. Since they depend only on the sufficient and complete statistics, they are UMVUE.

Example 6.35 (UMVUE may not attain Cramer-Rao lower bound)

Let $X \sim \text{Poisson}(\lambda)$. Let $T(X) = 1$ if $X = 0$ and $T(X) = 0$ otherwise. Then, T is UMVUE of $\tau(\lambda) = e^{-\lambda}$ and $\text{Var}(T) = e^{-\lambda}(1 - e^{-\lambda})$. The Cramer-Rao lower bound for $e^{-\lambda}$ is $e^{-2\lambda}/I(\lambda) = e^{-2\lambda}\lambda$. Hence $\text{Var}(T) - e^{-2\lambda}\lambda = e^{-\lambda}(1 - e^{-\lambda} - e^{-\lambda}\lambda) = e^{-\lambda}P(X \geq 2) > 0$ and Cramer-Rao lower bound is not attained.

by 3.

Example 6.36 (UMVUE of Poisson zero probability)

Suppose X_1, \dots, X_n is a sample from a $\text{Poisson}(\lambda)$ distribution. Then $S = \sum_{i=1}^n X_i = n\bar{X}$ is complete and sufficient for λ . To find a UMVUE of $e^{-\lambda}$, start with $I(X_1 = 0)$, which is an unbiased estimator. Since $I(X_1 = 0)$ is not a good estimator

$$E[I(X_1 = 0) | S = s] = 1 \cdot P(X_1 = 0 | S = s) = \frac{P(X_1 = 0, \sum_{i=2}^n X_i = s)}{P(\sum_{i=1}^n X_i = s)}$$

$$= \frac{e^{-\lambda} (e^{-(n-1)\lambda} [(n-1)\lambda]^s / s!)}{e^{-n\lambda} (n\lambda)^s / s!} = (1 - \frac{1}{n})^s$$

the UMVUE of $e^{-\lambda}$ is $(1 - n^{-1})^{n\bar{X}}$. Reasonable? $(1 + \frac{(-1)^n}{n})^{\bar{X}} \rightarrow e^{\bar{X}}$, when $n \rightarrow \infty$

Example 6.37 (cont. Ex. 6.25, UMVUE of Uniform upper bound, c.f. Ex. 6.13)

Suppose X_1, \dots, X_n is an i.i.d. sample from Uniform distribution $U(0, \theta)$. Because $X_{(n)}$ is sufficient and complete, and $E(X_{(n)}) = \frac{n}{n+1}\theta$, $\frac{n+1}{n}X_{(n)}$ is unbiased and is the UMVUE of θ .

Theorem 6.20 If $\rho_2 = g(\rho_1)$, then $\{S_2 = \rho_2\} \supseteq \{S_1 = \rho_1\}$ (check graph in Lnp.50)

Suppose that S_1 and S_2 are sufficient statistics for θ , and $S_2 = g(S_1)$. If U is an unbiased estimator for $\tau(\theta)$, let $V_1 = E(U|S_1)$ and $V_2 = E(U|S_2)$ then

$$V_2 = E[E(U|S_1)|S_2] = E(V_1|S_2) \Rightarrow \text{Var}(V_2) \leq \text{Var}(V_1)$$

Intuition: check graph in Lnp. 70. UMVUE is a function of minimal sufficient statistic

* Reading: textbook, 8.8.2; Further reading: Hogg et al., 7.1, 7.3, 7.6