Interval Estimation

cf. → Point Estimation

- What is interval estimation?

Question 6.10

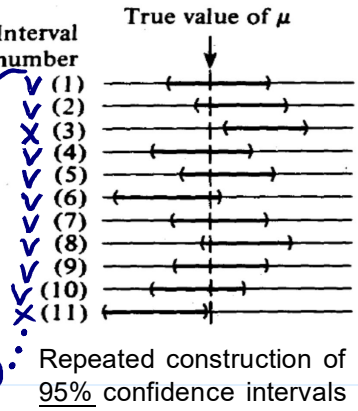
- Is it satisfactory to report only an estimated value of θ? → point estimation
- Note that
  1. A point estimate, although it will represent our best guess for the true value of the parameter, may be close to that true value but will virtually *never equal* it. e.g. $P(\hat{\Theta}=\Theta)=0$, if $\hat{\Theta}$ is a continuous r.v.
  2. Some measure of how close the point estimate is to the true value is required. One way to do this is to report both the *estimate* and its *estimated standard error*.
- The following questions arise naturally:
  1. A point estimator only gives a value. Wound it be better that we can give customer a range of possible values? This amounts to replacing the point estimate, a single value, by an entire interval of plausible values. → interval estimation
  2. Is there an estimation method that can combine together the two types of information, i.e., estimated value and estimated standard error?

Definition 6.22 (interval estimator, coverage probability, interval estimate, confidence interval, and confidence level, TBp. 217 & 279)

- For a random vector $\mathbf{X} = (X_1, \ldots, X_n)$, an **interval estimator** of a parameter θ with **coverage probability** $1-\alpha$ is a *random interval*
  $(\hat{\theta}_L(\mathbf{X}), \hat{\theta}_U(\mathbf{X}))$,
  where
  1. $\hat{\theta}_L(\mathbf{X})$, $\hat{\theta}_U(\mathbf{X})$ are functions of data,
  2. $\hat{\theta}_L(\mathbf{X}) < \hat{\theta}_U(\mathbf{X})$, and,
  3. $P(\theta \in (\hat{\theta}_L(\mathbf{X}), \hat{\theta}_U(\mathbf{X}))) \geq 1-\alpha$.

  data, treated as r.v.'s; assign $\mathbf{X}$ a joint dist.; $0<\alpha<1$; statistics (random variables)
  random fixed: $P(X \in (a,b))$; cf. 3.; Sometimes, defined as "$\geq$"; probability that the interval estimator will cover θ
- If $\mathbf{X} = \mathbf{x}$ is observed, the interval $(\hat{\theta}_L(\mathbf{x}), \hat{\theta}_U(\mathbf{x}))$ is called an **interval estimate**. (values; values)
- The term "$100 \cdot (1-\alpha)\%$ **confidence interval**" is used to denote either an interval estimator with coverage probability $1-\alpha$ or an interval estimate. (信賴區間; same; 信賴水準)
- The $100(1-\alpha)\%$ is also referred to as **confidence level**.
- **Note.** The α is usually assigned a small value, e.g. 0.1, 0.05, or 0.01. (90%, 95%, 99%)

95% correct on average



Question 6.11

How to interpret the $100(1-\alpha)\%$ in a $100(1-\alpha)\%$ interval *estimate* $(\hat{\theta}_L(\mathbf{x}), \hat{\theta}_U(\mathbf{x}))$? For example, for a 95% interval estimate, say $(23.5, 47.8)$, can we say that $P(\theta \in (23.5, 47.8)) = 0.95$? NO.

θ not random

the probability is either 0 or 1.

• methods for constructing interval estimators

Recall. methods for the estimation of standard error

LNp.12

When (i) sample size is large enough, (ii) difficult to get (1)

When (i) sample size is not large enough, and (ii) difficult to get (1)

(1) exact distribution (2) asymptotic method (3) bootstrap → sampling distribution

The three methods can be applied to construct confidence intervals.

Definition 6.23 (pivotal quantity)

A function of data $X_1, \ldots, X_n$ and parameter θ, denoted by $Q(\mathbf{X}, \theta) = Q(X_1, \ldots, X_n, \theta)$, is called a **pivotal quantity** for θ if the distribution of $Q(\mathbf{X}, \theta)$ is irrelevant to *all* parameters. (cf. ancillary statistics) ← LNp.52

e.g. $N(\mu, \sigma^2)$, $\theta = \mu$, all parameters: (μ, σ^2)

Note We may have more parameters in addition to θ.

cf. r.v. $Q(\mathbf{X}, \theta) = Q(X_1, \ldots, X_n, \theta)$, to construct C.I for θ

cf. not a statistic

Example 6.38 (some pivotal quantities, TBp. 279-281)

Poisson(λ) cf. mean $1/\lambda$

1. Let $X_1, \ldots, X_n$ be i.i.d. from Exponential distribution $E(\lambda)$, then

MLE of λ : $1/\bar{X}$

$\sum_{i=1}^n X_i \sim$ Gamma(n, λ)

$$\lambda \sum_{i=1}^{n} X_i \sim \Gamma(n, 1).$$

LN, CH1~6, p.74, items 4 & 6

unknown parameter

Because the pdf of $\Gamma(n,1)$ is irrelevant to λ, $\lambda \sum_{i=1}^n X_i$ is a pivotal quantity.

a function of data & λ

2. Let $X_1, \ldots, X_n$ be i.i.d. from Uniform distribution $U(0, \theta)$. Then, the pdf of is

$Y_i = \frac{X_i}{\theta}$ i.i.d. $U(0,1)$

$\frac{X_{(n)}}{\theta} = Y_{(n)} = \max\{Y_1, \cdots, Y_n\}$

$Q = \frac{X_{(n)}}{\theta}$

MLE of θ : $X_{(n)}$

unknown parameter

$$nq^{n-1}, \quad 0 \le q \le 1.$$

a function of data & θ

Because the pdf is irrelevant to θ, $X_{(n)}/\theta$ is a pivotal quantity.

3. Let $X_1, \ldots, X_n$ be i.i.d. from Normal distribution $N(\mu, \sigma^2)$, where μ is a parameter and the value of σ is known. Then,

unknown parameter

σ: not a parameter

MLE of μ : $\bar{X}$

$\bar{X} \sim N(\mu, \sigma^2/n)$ standardization

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0, 1).$$

LN, CH1~6, P.80, item 1

Because the pdf of $N(0,1)$ is irrelevant to μ, $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ is a pivotal quantity.

not a pivotal quantity for μ in case 4 (LNp.79)

a function of data & μ