by 2 — LNp 69, by 1

Example 6.33 (cont. Ex. 6.32, UMVUE of Poisson mean)

Suppose $X_1, \ldots, X_n$ is an i.i.d. sample from Poisson(λ) distribution. Then $S = \sum_{i=1}^n X_i$ is a sufficient and complete statistics for λ. Because $\overline{X} = S/n$ is a function of S and $E(\overline{X}) = \lambda$, $\overline{X}$ is UMVUE of λ. LNp.56 $\boxed{S \sim P(n\lambda) \Rightarrow E(S) = n\lambda}$

Example 6.34 (UMVUE of Normal mean and variance)

Let $X_1, \ldots, X_n$ be i.i.d. random variables from Normal distribution $N(\mu, \sigma^2)$, then

by 2. LNp.57

$$\overline{X} \text{ and } S^2 = \frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n-1}$$

are sufficient and complete statistics for (μ, σ^2). Clearly, $E(\overline{X}) = \mu$ and $E(S^2) = \sigma^2$ (**Note.** $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1} \Rightarrow E((n-1)S^2/\sigma^2) = n-1$), so $\overline{X}$ and S^2 are unbiased estimator of μ and σ^2, respectively. Since they depend only on the sufficient and complete statistics, they are UMVUE. LN.CHI-6. p.80

sufficient & complete $= P(X=0) = e^{-\lambda}\lambda^0/0! = e^{-\lambda}$ $\boxed{E(T) = e^{-\lambda}}$ $\boxed{T = I(X=0) \sim \text{Bernoulli}(e^{-\lambda})}$

Example 6.35 (UMVUE may not attain Cramer-Rao lower bound)

Let $X \sim$ Poisson(λ). Let $T(X) = 1$ if $X = 0$ and $T(X) = 0$ otherwise. Then, T is UMVUE of $\tau(\lambda) = e^{-\lambda}$ and $\text{Var}(T) = e^{-\lambda}(1 - e^{-\lambda})$. The Cramer-Rao lower bound for $e^{-\lambda}$ is $e^{-2\lambda}/I(\lambda) = e^{-2\lambda}\lambda$. Hence — by Thm.6.16 (LNp.65)

by 2

$$\text{Var}(T) - e^{-2\lambda}\lambda = e^{-\lambda}(1 - e^{-\lambda} - e^{-\lambda}\lambda) = e^{-\lambda}P(X \geq 2) > 0$$

and Cramer-Rao lower bound is not attained. $\boxed{X \xrightarrow{\text{?}} \lambda\text{, so, use } e^{-X} \xrightarrow{\text{?}} e^{-\lambda}\text{?}}$

by 3.

Example 6.36 (UMVUE of Poisson zero probability)

Suppose $X_1, \ldots, X_n$ is a sample from a Poisson(λ) distribution. Then $S = \sum_{i=1}^n X_i = n\overline{X}$ is complete and sufficient for λ. To find a UMVUE of $e^{-\lambda}$, start with $I(X_1 = 0)$, which is an unbiased estimator. Since $\sim P(\lambda)$ $\sim \text{Poisson}((n-1)\lambda)$

not a good estimator

$$\text{E}[I(X_1 = 0)|S = s] \;=\; 1 \cdot P(X_1 = 0|S = s) = \frac{P(X_1 = 0, \sum_{i=2}^n X_i = s)}{P(\sum_{i=1}^n X_i = s)}$$

$\sim \text{Poisson}(n\lambda)$

MLE: $e^{-\overline{X}}$ $\Rightarrow$ not unbiased — unbiased $\boxed{\sum_{i=1}^n I(X_i = 0)/n}$ c.f. — by 2 — c.f.

$$= \frac{\cancel{e^{-\lambda}}(\cancel{e^{-(n-1)\lambda}})[(n-1)\cancel{\lambda}]^s/\cancel{s!}}{\cancel{e^{-n\lambda}}(n\cancel{\lambda})^s/\cancel{s!}} = (1 - \frac{1}{n})^s,$$

the UMVUE of $e^{-\lambda}$ is $(1 - n^{-1})^{n\overline{X}}$. ← reasonable? $([1 + \frac{(-1)}{n}]^n)^{\overline{X}} \xrightarrow{P} \lambda$ ($\nearrow e^{-1}$), when $n \to \infty$

Example 6.37 (cont. Ex. 6.25, UMVUE of Uniform upper bound, c.f. Ex. 6.13)

Suppose $X_1, \ldots, X_n$ is an i.i.d. sample from Uniform distribution $U(0, \theta)$. Because $X_{(n)}$ is sufficient and complete, and $E(X_{(n)}) = \frac{n}{n+1}\theta$, $\frac{n+1}{n}X_{(n)}$ is unbiased and is the UMVUE of θ. LNp.53, Ex.6.25 | (LNp.22) moment est'or: $2\overline{X}$, MLE: $X_{(n)}$ — c.f.

Theorem 6.20 If $\mathcal{B}_2 = g(\mathcal{B}_1)$, then $\{S_2 = \mathcal{B}_2\} \supseteq \{S_1 = \mathcal{B}_1\}$ (check graph in LNp 50)

Suppose that S_1 and S_2 are sufficient statistics for θ, and $S_2 = g(S_1)$. If U is an unbiased estimator for $\tau(\theta)$, let $V_1 = E(U|S_1)$ and $V_2 = E(U|S_2)$ then

$V_2 = E[V_1|S_2] = E[E[U|S_1]|S_2]$ | Intuition: check graph in LNp. 70 →

$$\text{Var}(V_2) \leq \text{Var}(V_1).$$

→ UMVUE is a function of minimal sufficient statistic

❖ **Reading**: textbook, 8.8.2; **Further reading**: Hogg et al., 7.1, 7.3, 7.6