Definition 5.1 (converge almost surely, TBp. 178)

A sequence of random variables $\{Z_n : \Omega \to \mathbb{R}\}$ is said to **converge almost surely** to a random variable $Z : \Omega \to \mathbb{R}$, and denoted as $Z_n \xrightarrow{\text{a.s.}} Z$, if for any $\epsilon > 0$,

$$P\left(\left\{\omega \in \Omega : \lim_{n\to\infty} |Z_n(\omega) - Z(\omega)| < \epsilon\right\}\right) = 1.$$

Z_n alike Z as $n\to\infty$; converge pointwise; $P(A)=1$, $P(A^c)=0$; $\lim_{n\to\infty}|Z_n(\omega)-Z(\omega)| = 0 \Leftrightarrow \lim_{n\to\infty} Z_n(\omega) = Z(\omega)$

Definition 5.2 (converge in probability, TBp. 178)

A sequence of random variables $\{Z_n : \Omega \to \mathbb{R}\}$ is said to **converge in probability** to a random variable $Z : \Omega \to \mathbb{R}$, and denoted as $Z_n \xrightarrow{P} Z$, if for any $\epsilon > 0$,

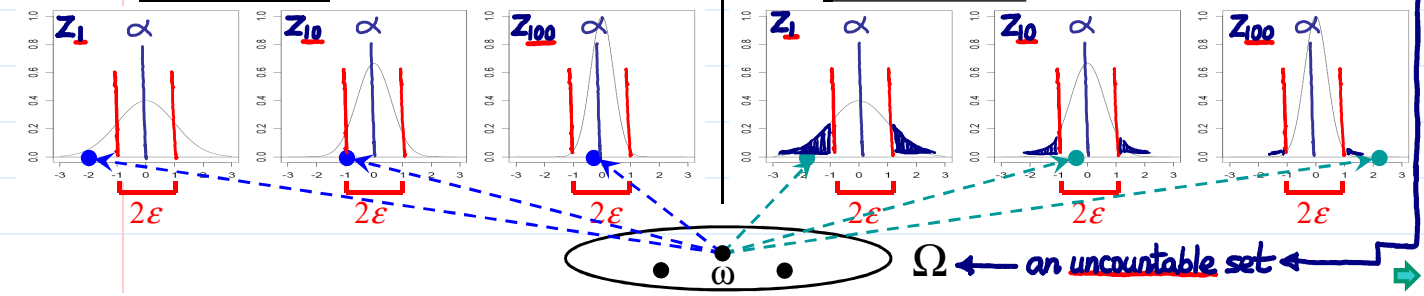
$$\lim_{n\to\infty} P\left(\{\omega \in \Omega : |Z_n(\omega) - Z(\omega)| < \epsilon\}\right) = 1.$$

A_n; $P(\cdot)$ on $A_1, A_2, \ldots, A_n, \ldots \to 1$; ω: infinitely many times; cont. r.v., $P(\{\omega\})=0$

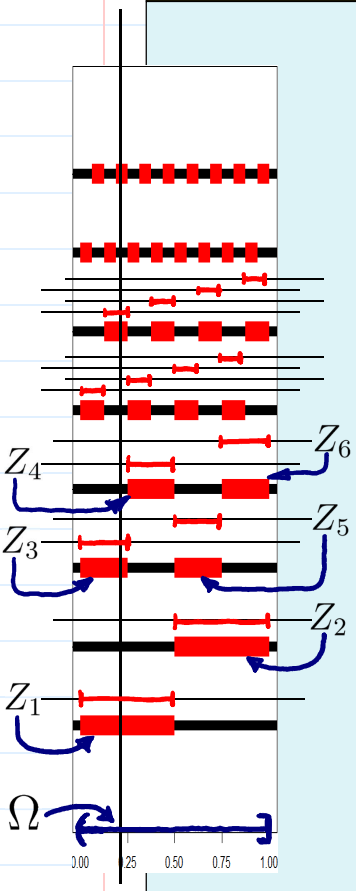
cf. $Z=\alpha$

$Z_n \xrightarrow{a.s.} \alpha$, α : a constant. ($Z_n \xrightarrow{a.s.} Z \Leftrightarrow Y_n = Z_n - Z \xrightarrow{a.s.} 0$)

$Z_n \xrightarrow{P} \alpha$, α : a constant. ($Z_n \xrightarrow{P} Z \Leftrightarrow Y_n = Z_n - Z \xrightarrow{P} 0$)



Example 5.1 (convergence in probability need not imply convergence a.s.)



- $\Omega = (0, 1]$
- P: uniform probability measure on Ω ← i.e., $0<a<b<1$, $P([a,b]) = b-a$
- For $k = 1, 2, \ldots,$ divide $(0, 1]$ into 2^k subintervals of equal length. These intervals are given by
$$I_{k,j} = \left(\frac{j-1}{2^k}, \frac{j}{2^k}\right] \to P(I_{k,j}) = \frac{1}{2^k}$$
for $j = 1, 2, \ldots, 2^k$.
- Let $Z_1, Z_2, \cdots : \Omega \to \mathbb{R}$ be a sequence of r.v.'s defined as follows:
$$Z_n(\omega) = \begin{cases} 1, & \text{if } \omega \in I_{k,j}, \\ 0, & \text{if } \omega \notin I_{k,j}, \end{cases}$$
where $n = 2^k + j - 2$.
- $Z : \Omega \to \mathbb{R}$ such that $Z(\omega) = 0$ for $\omega \in \Omega$
- $Z_n \xrightarrow{P} Z$, because for $0 < \epsilon < 1$,
$$P\left(\{\omega \in \Omega : |Z_n(\omega) - Z(\omega)| < \epsilon\}\right) = 1 - \frac{1}{2^k} \to 1.$$
- Z_n not converge to Z a.s. because
$$P\left(\{\omega \in \Omega : Z_n(\omega) \to Z(\omega)\}\right) = P(\emptyset) = 0.$$

Definition 5.3 (converge in distribution, TBp. 181)

Let $Z_1, Z_2, \ldots$ be a sequence of random variables with cdf's $F_1, F_2, \ldots$ and let Z be a random variable with cdf F. Then Z_n **converges in distribution** to Z, denoted as $Z_n \xrightarrow{d} Z$, if

$$\lim_{n\to\infty} F_n(z) = F(z)$$

at every point z where F is continuous.

Theorem 5.1 (some properties about the 3 types of convergence)

1. $Z_n \xrightarrow{\text{a.s.}} Z \Rightarrow Z_n \xrightarrow{P} Z$
2. $Z_n \xrightarrow{P} Z \Rightarrow Z_n \xrightarrow{d} Z$
3. $Z_n \xrightarrow{d} c$, c: a constant $\Rightarrow Z_n \xrightarrow{P} c$
4. **(convergence of transformation)** Let $g : \mathbb{R}^k \mapsto \mathbb{R}$ be a continuous function.
   (a) $Z_n^{(j)} \xrightarrow{\text{a.s.}} Z^{(j)}, j = 1, \ldots, k \Rightarrow g(Z_n^{(1)}, \ldots, Z_n^{(k)}) \xrightarrow{\text{a.s.}} g(Z^{(1)}, \ldots, Z^{(k)})$.
   (b) $Z_n^{(j)} \xrightarrow{P} Z^{(j)}, j = 1, \ldots, k \Rightarrow g(Z_n^{(1)}, \ldots, Z_n^{(k)}) \xrightarrow{P} g(Z^{(1)}, \ldots, Z^{(k)})$.
   (c) $(Z_n^{(1)}, \ldots, Z_n^{(k)}) \xrightarrow{d} (Z^{(1)}, \ldots, Z^{(k)}) \Rightarrow g(Z_n^{(1)}, \ldots, Z_n^{(k)}) \xrightarrow{d} g(Z^{(1)}, \ldots, Z^{(k)})$.

5. **(Slutsky's theorem)** If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a$, where a is a constant, then
   (a) $Y_n X_n \xrightarrow{d} aX$
   (b) $X_n + Y_n \xrightarrow{d} X + a$
   (c) $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{a}$, provided that $P(Y_n \neq 0) = 1$ for all n and $a \neq 0$.

6. **(limit theorem for δ method)** Suppose

$$\frac{\sqrt{n}(X_n - \theta)}{\sigma} \xrightarrow{d} N(0,1).$$

For a given function g, suppose that $g'(\theta) \neq 0$ exists. Then

$$\frac{\sqrt{n}[g(X_n) - g(\theta)]}{\sigma|g'(\theta)|} \xrightarrow{d} N(0,1).$$