(A1, B1) (*14pts, 2pts for each*)

Exam A.
(a) False (b) True (c) True (d) False
(e) False (f) False (g) True

Exam B.
(a) True (b) False (c) False (d) False
(e) True (f) False (g) True

(A2, B2) (*18pts, 6pts for each*)

Exam A.

(a) **(1)** $X_i \sim \text{binomial}(200, p_i)$, $i = 1, \ldots, 4$, and $X_1, \ldots, X_4$ are independent
(2) $\Omega = \{(p_1, \ldots, p_4) |\ 0 < p_i < 1,\ i = 1, \ldots, 4\}$. (Note that $\dim(\Omega) = 4$, not 3.)
(3) $\Omega_0 = \{(p_1, \ldots, p_4) |\ 0 < p_1 = \cdots = p_4 < 1\}$. (Note that $\dim(\Omega_0) = 1$.)

(b) **(1)** $X_1, \ldots, X_8$ are i.i.d. from normal(μ, σ^2)
(2) $\Omega = \{(\mu, \sigma) |\ -\infty < \mu < \infty,\ 0 < \sigma < \infty\}$
(3) $\Omega_0 = \{(\mu, \sigma) |\ \mu = 0,\ 0 < \sigma < \infty\}$

(c) **(1)** $X_1, \ldots, X_{20}$ are i.i.d. from Bernoulli(p), where p is the probability that an animal receiving the drug benefits more than its mate.
(2) $\Omega = \{p : 1/2 \le p < 1\}$. The Ω assumes that it is impossible for the drug to have a negative effect.
(3) $\Omega_0 = \{p : p = 1/2\}$. If the null is rejected, it would be concluded that the drug has a positive effect.

Exam B.

(b) **(1)** $X_i \sim \text{binomial}(300, p_i)$, $i = 1, \ldots, 3$, and $X_1, \ldots, X_3$ are independent
(2) $\Omega = \{(p_1, \ldots, p_3) |\ 0 < p_i < 1,\ i = 1, \ldots, 3\}$. (Note that $\dim(\Omega) = 3$, not 2.)
(3) $\Omega_0 = \{(p_1, \ldots, p_3) |\ 0 < p_1 = \cdots = p_3 < 1\}$. (Note that $\dim(\Omega_0) = 1$.)

(c) **(1)** $X_1, \ldots, X_{10}$ are i.i.d. from normal(μ, σ^2)
(2) $\Omega = \{(\mu, \sigma) |\ -\infty < \mu < \infty,\ 0 < \sigma < \infty\}$
(3) $\Omega_0 = \{(\mu, \sigma) |\ \mu = 0,\ 0 < \sigma < \infty\}$

(a) **(1)** $X_1, \ldots, X_{30}$ are i.i.d. from Bernoulli(p), where p is the probability that an animal receiving the drug benefits more than its mate.
(2) $\Omega = \{p : 1/2 \le p < 1\}$. The Ω assumes that it is impossible for the drug to have a negative effect.
(3) $\Omega_0 = \{p : p = 1/2\}$. If the null is rejected, it would be concluded that the drug has a positive effect.

(A3, B6) (*16pts*)

(a) (*2pts*) For $\theta < y < \infty$,

$$\begin{aligned} F_Y(y|\theta) &= P(Y \le y) = 1 - P(X_{(1)} > y) = 1 - P(X_1 > y, \ldots, X_n > y) \\ &= 1 - \prod_{i=1}^{n} P(X_i > y) = 1 - \left[e^{-(y-\theta)}\right]^n = 1 - e^{-n(y-\theta)}. \end{aligned}$$

(b) (*4pts*) For $0 < t < \infty$, the cdf of $T_n(\theta)$ is:

$$P\left(T_n(\theta) \le t\right) = P\left(2n(Y - \theta) \le t\right) = P\left(Y \le \frac{t}{2n} + \theta\right) = 1 - e^{-n\left(\frac{t}{2n} + \theta - \theta\right)} = 1 - e^{-\frac{t}{2}},$$

which is irrelevant to θ. Because $T_n(\theta)$ is a function of parameter θ and data $X_1, \ldots, X_n$, and the distribution of $T_n(\theta)$ is irrelevant to θ, $T_n(\theta)$ is a pivotal quantity.

[**Note**: The distribution of $T_n(\theta)$ is exponential(1/2) (or gamma(1, 1/2), or chi-square χ^2_2 distribution).]

(c) (*6pts*) Let a and b satisfy:

$$\frac{\alpha}{2} = P\left(T_n(\theta) \le a\right) = 1 - e^{-\frac{a}{2}} \qquad \Rightarrow \qquad a = -2\log\left(1 - \frac{\alpha}{2}\right),$$

and

$$\frac{\alpha}{2} = P\left(T_n(\theta) > b\right) = e^{-\frac{b}{2}} \qquad \Rightarrow \qquad b = -2\log\left(\frac{\alpha}{2}\right).$$

Then, from (b), we know

$$1 - \alpha = P\left(a \le T_n(\theta) = 2n(Y - \theta) \le b\right) = P\left(Y - \frac{b}{2n} \le \theta \le Y - \frac{a}{2n}\right).$$

Therefore,

$$\left[Y + \frac{\ln(\alpha/2)}{n}, Y + \frac{\ln(1-\alpha/2)}{n}\right]$$

is a $100(1-\alpha)\%$ confidence interval for θ.

(d) (*4pts*) To construct a two-sided level-α test, we reject H_0 whenever θ_0 does not belong to the confidence interval in (c), i.e.,

$$\theta_0 \notin \left[Y + \frac{\ln(\alpha/2)}{n},\ Y + \frac{\ln(1-\alpha/2)}{n}\right].$$

Thus, the rejection region is

$$Y - \theta_0 < -\frac{\ln(1-\alpha/2)}{n} \ \text{ or } \ Y - \theta_0 > -\frac{\ln(\alpha/2)}{n},$$

or equivalently,

$$Y < \theta_0 - \frac{\ln(1-\alpha/2)}{n} \ \text{ or } \ Y > \theta_0 - \frac{\ln(\alpha/2)}{n}.$$

(A4, B3) (*16pts*)

(a) (*12pts*) The joint pdf of $X_1, \ldots, X_n$ is

$$f(\mathbf{x}|\sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-x_i^2/(2\sigma^2)\right) = \left(\sqrt{2\pi}\sigma\right)^{-n} \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2\sigma^2}\right).$$

Therefore, the likelihood ration Λ is

$$\Lambda = \frac{f(\mathbf{x}|\sigma_0)}{f(\mathbf{x}|\sigma_1)} = \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left[\left(\sum_{i=1}^n x_i^2\right)\left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}\right)\right],$$

where $\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2} < 0$ when $\sigma_1 > \sigma_0$. Because $\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2} < 0$, Λ decreases as $\sum_{i=1}^n X_i^2$ increases. We should reject H_0 when $\sum_{i=1}^n X_i^2$ is large (i.e., larger $\sum_{i=1}^n X_i^2$ is more extreme). The test function based on the likelihood ratio is then given by:

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n X_i^2 > c, \\ 0, & \text{if } \sum_{i=1}^n X_i^2 \le c. \end{cases}$$

The c is determined by

$$\mathrm{E}_{\sigma_0}(\phi) = P_{\sigma_0}\left(\sum_{i=1}^{n} X_i^2 > c\right) = P_{\sigma_0}\left(\sum_{i=1}^{n} X_i^2/\sigma_0^2 > c/\sigma_0^2\right) = \alpha.$$

Because $\sum_{i=1}^{n} X_i^2/\sigma_0^2 \sim \chi_n^2$ under H_0, we can get $c = \sigma_0^2\chi_n^2(1-\alpha)$, where $\chi_n^2(1-\alpha)$ is the $(1-\alpha)$-quantile of the χ_n^2 distribution.

(b) (*4pts*) By Neyman-Pearson lemma, the test in (a) is the most powerful test for any particular simple alternative $H_A : \sigma = \sigma_1$, where $\sigma_1 > \sigma_0$. Furthermore, because the rejection region and the critical value (i.e., c) of the test does not depend on σ_1, the test is UMP for $H_0 : \sigma = \sigma_0$ versus $H_A : \sigma > \sigma_0$.

(A5, B4) (*20 pts*)

(a) (*3pts*) Under hardy-Weinberg equilibrium, $(X_1, X_2, X_3) \sim$ multinomial$(n, p_1(\theta), p_2(\theta), p_3(\theta))$ with $n = 190$.

(b) (*3pts*) For testing $H_0 : \theta = \frac{1}{2}$ versus $H_A : \theta \neq \frac{1}{2}$,

$$\Omega = \{\theta : 0 < \theta < 1\} \quad \text{and} \quad \Omega_0 = \left\{\frac{1}{2}\right\} \quad \Rightarrow \quad \dim(\Omega) = 1 \quad \text{and} \quad \dim(\Omega_0) = 0.$$

(c) (*5pts*) The likelihood function of θ given $\mathbf{X} = (X_1, X_2, X_3)$ is

$$\mathcal{L}(\theta|\mathbf{x}) = \frac{n!}{x_1!x_2!x_3!} p_1(\theta)^{x_1} p_2(\theta)^{x_2} p_3(\theta)^{x_3}.$$

The generalized likelihood ratio test statistic Λ is

$$\Lambda = \frac{\max_{\theta\in\Omega_0} \mathcal{L}(\theta|\mathbf{x})}{\max_{\theta\in\Omega} \mathcal{L}(\theta|\mathbf{x})} = \prod_{i=1}^{3} \left(\frac{p_i(\theta_0)}{p_i(\hat{\theta}_{MLE})}\right)^{x_i},$$

where $\theta_0 = \frac{1}{2}$ is from Ω_0 and $\hat{\theta}_{MLE} = \frac{2X_3+X_2}{2n}$ is the MLE of θ under Ω.

(d) (*3pts*) Because $\dim(\Omega) - \dim(\Omega_0) = 1$, the null distribution of $-2\log\Lambda$ tends to a χ_1^2 distribution when n is large.

(e) (*6pts*) For these data, the test statistic gives

$$\begin{aligned}
-2\log\Lambda &= -2\sum_{i=1}^{3} x_i \log\left(\frac{p_i(\theta_0)}{p_i(\hat{\theta}_{MLE})}\right). \\
&= -2\left[10\cdot\log\left(\frac{p_1(0.5)}{p_1(0.77)}\right) + 68\cdot\log\left(\frac{p_2(0.5)}{p_2(0.77)}\right) + 112\cdot\log\left(\frac{p_3(0.5)}{p_3(0.77)}\right)\right] \\
&= -2\left[10\cdot 1.55 + 68\cdot 0.34 + 112\cdot(-0.86)\right] = 115.4
\end{aligned}$$

The critical value for $\alpha = 0.05$ is $\chi_1^2(0.05) = 3.841$. Therefore, H_0 is rejected if $-2\log\Lambda > \chi_1^2(0.05)$. It is very clear that the data rejects the H_0 overwhelmingly.

(A6, B5) (*16pts*)

(a) (*5pts*) When the prior distribution of p is beta(α, β), the posterior pdf of $p|x_1, \ldots, x_n$ is

$$\begin{aligned} h(p|x_1, \ldots, x_n) &\propto f(x_1, \ldots, x_n|p) \cdot g(p) = \left[\prod_{i=1}^{n} f(x_i|p)\right] \cdot g(p) \\ &\propto \left[\prod_{i=1}^{n} p(1-p)^{x_i-1}\right] \cdot p^{\alpha-1}(1-p)^{\beta-1} \\ &= p^n(1-p)^{(\sum_{i=1}^n x_i)-n} \cdot p^{\alpha-1}(1-p)^{\beta-1} \\ &= p^{(n+\alpha)-1}(1-p)^{[(\sum_{i=1}^n x_i)-n+\beta]-1}, \end{aligned}$$

which follows the form of the beta pdf with parameters $n+\alpha$ and $(\sum_{i=1}^n x_i) - n + \beta$.

(b) (*4pts*) Because $p|x_1, \ldots, x_n \sim$ beta$(n+\alpha, (\sum_{i=1}^n x_i)-n+\beta)$, the Bayes estimator under squared error loss is

$$\begin{aligned} \hat{p}_B &= E\left(p|x_1, \ldots, x_n\right) \\ &= \frac{n+\alpha}{(n+\alpha) + ((\sum_{i=1}^n x_i) - n + \beta)} = \frac{n+\alpha}{(\sum_{i=1}^n x_i) + \alpha + \beta} \\ &= \underbrace{\frac{\alpha+\beta}{(\sum_{i=1}^n x_i) + \alpha + \beta}}_{w_1} \times \frac{\alpha}{\alpha+\beta} + \underbrace{\frac{\sum_{i=1}^n x_i}{(\sum_{i=1}^n x_i) + \alpha + \beta}}_{w_2} \times \frac{n}{\sum_{i=1}^n x_i}, \end{aligned} \tag{I}$$

where $\frac{\alpha}{\alpha+\beta}$ is the prior mean, $\frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\overline{X}}$ is the MLE $\hat{p}_{MLE}$, and (w_1, w_2) are weights because $w_1 + w_2 = 1$.

(c) (*2pts*) We can write (I) as

$$\hat{p}_B = \frac{1 + \frac{\alpha}{n}}{\overline{X} + \frac{\alpha+\beta}{n}}. \tag{II}$$

When n is large, because α and β are fixed values, α/n and $(\alpha+\beta)/n$ are close to zero and have little influence on the value of the Bayes estimator. Under the circumstance, the major component that determines the value of the Bayes estimator is $\overline{X}$, which comes from the sample.

(d) (*2pts*) By law of large number, $\overline{X}$ converge in probability to $1/p$, the mean of geometric(p). Because $\alpha/n \to 0$ and $(\alpha+\beta)/n \to 0$ when $n \to \infty$, applying the continuous mapping theorem on (II) gives

$$\hat{p}_B \xrightarrow{P} \frac{1+0}{1/p+0} = p.$$

(e) (*3pts*) The estimator $\tilde{\theta}$ is the Bayes estimator when $\alpha = 1$ and $\beta = 1$ (i.e., prior $\sim$ uniform(0, 1)). Because a Bayes estimator with respect to proper prior is admissible, $\tilde{\theta}$ is an admissible estimator of p, and there is no estimator that can strictly dominate $\tilde{\theta}$ in terms of the risk function.