Note. There are 6 problems in total. **For problems 3 to 6, to ensure consideration for partial scores, write down necessary intermediate steps.** Correct answers with inadequate or no intermediate steps may result in zero credit.

Some useful formula.

- Suppose that X follows a binomial(n, p) distribution. The probability mass function (pmf) of X is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, \ldots, n.$$

The mean of X is np, and the variance of X is $np(1-p)$.

- Suppose that $(X_1, \ldots, X_m)$ follows a multinomial$(n, p_1, \ldots, p_m)$ distribution, where $p_1 + \cdots + p_m = 1$. The joint pmf of $(X_1, \ldots, X_m)$ is

$$p(x) = \binom{n}{x_1 \cdots x_m} p_1^{x_1} \cdots p_m^{x_m},$$

for $x_i = 0, \ldots, n$, where $i = 1, \ldots, m$, and $x_1 + \cdots + x_m = n$. For $i = 1, \ldots, m$, the marginal distribution of X_i is binomial(n, p_i). For $1 \le i < j \le m$, $\text{cov}(X_i, X_j) = -np_ip_j$.

- Suppose that X follows a Poisson(λ) distribution. The pmf of X is

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \text{ for } x = 0, 1, 2, \ldots.$$

The mean of X is λ, and the variance of X is λ.

- Suppose that X follows a uniform(a, b) distribution. The probability density function (pdf) of X is

$$f(x) = \frac{1}{b-a}, \text{ for } a < x < b,$$

and zero otherwise. The mean of X is $\frac{a+b}{2}$, and the variance of X is $\frac{(b-a)^2}{12}$.

- Suppose that X follows an exponential(λ) distribution. The pdf of X is

$$f(x) = \lambda e^{-\lambda x}, \text{ for } 0 < x < \infty,$$

and zero otherwise. The mean of X is $1/\lambda$, and the variance of X is $1/\lambda^2$.

- Suppose that X follws a gamma(α, λ) distribution. The pdf of X is

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \text{ for } 0 < x < \infty,$$

and zero otherwise. The mean of X is α/λ, and the variance of X is α/λ^2.

1. (*14 pts in total. for each question, 2 pts if correct; 0 pt if blank; −1 pt if wrong*) For the following statements, please answer true or false.

   (a) The statement, "a 95% confidence interval (i.e., an interval estimat*e*) for the parameter μ is (350, 400)", is equivalent to the statement, "the probability that μ falls in (350, 400) is 95%".

   (b) If you repeat the procedure of collecting data to generate a 95% confidence interval for μ over and over again, about 95% of the intervals should contain μ.

   (c) A 95% confidence interval (i.e., an interval estimat*or*) of μ obtained from a random sample of size 1000 (i.e., 1000 independent observations) has a better chance of containing μ than a 95% confidence interval obtained from a random sample of size 500.

   (d) For a test statistic, when the critical value is adjusted to increase the probability of type I error (i.e., α), the probability of type II error (i.e., β) always decreases or remains the same.

   (e) Consider a test function ϕ for testing $H_0 : \theta \in \Omega_0$ versus $H_A : \theta \in \Omega_A$. For $\theta \in \Omega_0$, the mean function of ϕ, i.e., $\mathrm{E}_\theta(\phi)$, gives the probability that the null hypothesis is falsely rejected.

   (f) For the hypotheses testing problem, the Neyman-Pearson approach suggests to minimize the total probability of type I and type II errors (i.e., $\alpha + \beta$).

   (g) Suppose that (i) the null hypothesis H_0 is simple, (ii) the test statistic T is a continuous random variable, and (iii) the test rejects for large values of T. The p-value of the test is a random variable with uniform$(0, 1)$ distribution under H_0.

2. (*12pts, 6pts for each*) For each of the random observations $\mathbf{X}$ below, determine (i) a joint distribution which best models $\mathbf{X}$. Then, formulate the problem of interest into a hypotheses testing problem by specifying for the distribution of $\mathbf{X}$ (ii) the space Ω of all possible parameters, and (iii) the space Ω_0 of parameters under null hypothesis.

   (a) Consider a basic experiment of tossing five coins at a time. It was known that the five coins had the same unknown probability, denoted by p, of getting heads. Let $q_i(p)$, $i = 0, 1, \ldots, 5$, denote the probability that a binomial$(5, p)$ random variable takes the value i. The basic experiment was repeated 100 times. In the 100 trials, the number of trials that had i heads, $i = 0, 1, \ldots, 5$, was recorded and denoted by X_i. Use the data $\mathbf{X} = (X_0, X_1, \ldots, X_5)$ to examine whether all the five coins were fair.

   (b) In a study to assess various effects of using a female model in automobile advertising, each of 100 randomly chosen male subjects was shown photographs of two automobiles matched for price, color, and size but of different makes. Fifty of the subjects (group A) were shown automobile 1 with a female model and automobile 2 with no model. Both automobiles were shown without the model to the other 50 subjects (group B). In group A, let X_A be the number of subjects who judged automobile 1 to be more expensive. In group B, let X_B be the number of subjects who judged automobile 1 to be more expensive.

Use the data $\mathbf{X} = (X_A, X_B)$ to examine the claim that using a female model can *increase* the perceived cost of an automobile.

3. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) random observations from the uniform$(0, \theta)$ distribution. Let $X_{(n)} = \max\{X_1, \ldots, X_n\}$. Consider testing

$$H_0 : \theta = 1 \text{ versus } H_A : \theta = 2.$$

   (a) (*2pts*) Are H_0 and H_A simple or composite?

   (b) (*4pts*) Find the most powerful test with significance level $\alpha = 0$. What is the power of the test?
   [**Hint**. (i) The joint pdf of $X_1, \ldots, X_n$ is $f(x_1, \ldots, x_n \mid \theta) = \theta^{-n} \cdot I_{[0,\theta]}(x_{(n)})$. (ii) The pdf and cdf of $X_{(n)}$ are given by:

$$f_{X_{(n)}}(x) = \frac{nx^{n-1}}{\theta^n}, \quad \text{for } 0 \leq x \leq \theta,$$
$$F_{X_{(n)}}(x) = \left(\frac{x}{\theta}\right)^n, \quad \text{for } 0 \leq x \leq \theta.\Big]$$

   (c) (*2pts*) For $0 < a < 1$, what is the significance level and power of the non-randomized test that rejects H_0 when $X_{(n)} \in [a, 2]$?

   (d) (*3pts*) Consider the randomized test that rejects H_0 if $X_{(n)} > 1$ and rejects H_0 with probability γ, $0 < \gamma < 1$, when $X_{(n)} \leq 1$. What is its significance level and power?

   (e) (*4pts*) For $0 < \alpha < 1$, is the most powerful test with significance level α unique? Explain.

4. Let $X_1, \ldots, X_n$ be i.i.d. from a Poisson(λ) distribution.

   (a) (*10 pts*) Find the likelihood ratio test for testing

$$H_0 : \lambda = \lambda_0 \quad \text{versus} \quad H_A : \lambda = \lambda_1,$$

   where $\lambda_1 > \lambda_0$. Use the fact that the sum of independent Poisson random variables follows a Poisson distribution, i.e.,

$$\sum_{i=1}^{n} X_i \sim \text{Poisson}(n\lambda),$$

   to explain how to determine a rejection region for a *randomized* test at level α. **Your final answer must at least include (i) a test function based on a test statistic, (ii) a critical value, (iii) an equation for determining the critical value, and (iv) the exact null distribution of the test statistic.**

   (b) (*4 pts*) Show that the test in (a) is uniformly most powerful (UMP) for testing

$$H_0 : \lambda = \lambda_0 \quad \text{versus} \quad H_A : \lambda > \lambda_0.$$

5. Suppose that $X_1, \ldots, X_n$ are i.i.d. from the uniform$(0, \theta)$ distribution, where $\theta \in \Omega = (0, \infty)$. Let $X_{(n)} = \max\{X_1, \ldots, X_n\}$. Then, the joint pdf of $X_1, \ldots, X_n$ can be written as:

$$f(x_1, \ldots, x_n \mid \theta) = \frac{1}{\theta^n} I_{[0,\theta]}(x_{(n)}), \tag{1}$$

where $x_{(n)} = \max\{x_1, \ldots, x_n\}$ and I is the indicator function, i.e.,

$$I_{[0,\theta]}(t) = \begin{cases} 1, & \text{if } t \in [0, \theta], \\ 0, & \text{otherwise.} \end{cases}$$

(a) (*7 pts*) For a fixed $\theta_0 \in (0, \infty)$, use (1) to show that the generalized likelihood ratio (GLR) Λ for testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_A : \theta \neq \theta_0$$

is

$$\Lambda(x_1, \ldots, x_n) = \begin{cases} \left(\dfrac{x_{(n)}}{\theta_0}\right)^n, & \text{if } 0 \leq x_{(n)} \leq \theta_0, \\ 0, & \text{if } x_{(n)} > \theta_0. \end{cases} \tag{2}$$

[**Hint:** (i) When $0 < \theta < \infty$, the MLE of θ is $\hat{\theta} = X_{(n)}$. (ii) $I_{[0,x_{(n)}]}(x_{(n)}) = 1$.]

(b) (*4 pts*) Use (2) to show that

$$\Lambda(X_1, \ldots, X_n) > s,$$

where $0 < s < 1$, is equivalent to

$$s^{1/n}\theta_0 < X_{(n)} < \theta_0.$$

[**Hint:** draw the graph of Λ as a function of $x_{(n)}$ to help you identify the region.]

(c) (*8 pts*) Use (b) to derive the rejection region of the GLR test at the significance level $\alpha = 0.05$. **Please express the rejection region in terms of $X_{(n)}$ and find the value of s for $\alpha = 0.05$.**
[**Hint:** The pdf and cdf of $X_{(n)}$ are given by:

$$f_{X_{(n)}}(x) = \frac{nx^{n-1}}{\theta^n}, \quad \text{for } 0 \leq x \leq \theta,$$
$$F_{X_{(n)}}(x) = \left(\frac{x}{\theta}\right)^n, \quad \text{for } 0 \leq x \leq \theta.]$$

(d) (*4 pts*) Use the acceptance region of the GLR test in (c) to construct a 95% confidence interval for θ.

(e) (*5 pts*) Use your answer to (d) to identify a pivotal quantity for θ and explain why it is a pivotal quantity.

6. Let $X_1, \ldots, X_n$ be an i.i.d. sample from the exponential(θ) distribution. Let θ have a prior distribution gamma(α, λ), with known $\alpha > 0$, $\lambda > 0$. Let $\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Notice that the MLE of θ is $\hat{\theta}_{\text{MLE}} = 1/\overline{X_n}$.

   (a) (*5 pts*) Show that the posterior distribution of θ given the data is

   $$\theta \mid X_1, \ldots, X_n \sim \text{gamma}(\alpha + n, \lambda + n\overline{X_n}).$$

   (b) (*5 pts*) Show that the Bayes estimator of θ under *squared error loss* is

   $$\hat{\theta}_B = \frac{\alpha + n}{\lambda + n\overline{X_n}},$$

   and show that it is a weighted average of the prior mean and $\hat{\theta}_{\text{MLE}}$.

   (c) (*3 pts*) Use the result from (b) to show that when the sample size n is large, the information from the sample will dominate the estimation.

   (d) (*4 pts*) Is it possible for the $\hat{\theta}_{\text{MLE}}$ to strictly dominate $\hat{\theta}_B$ in terms of the risk function under squared error loss? Explain.