CH9

#1. $H_0: p=0.5$, $H_a: p\neq 0.5$

a. 令X為出現正面次數，在H_0之下，$p=0.5$

$X\sim Bin(10,0.5)$ and the test rejects if $x=0$ or $x=10$

$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true})$

$= P(X=0\mid p=0.5) + P(X=10\mid p=0.5)$

$= \binom{10}{0}0.5^0 0.5^{10} + \binom{10}{10}0.5^{10}0.5^0$

$= 2\times 0.5^{10} = \frac{2}{1024} = \frac{1}{512} \approx 0.00195$

b. $\text{power} = P(\text{reject } H_0 \mid p=0.1)$

$= P(X=0\mid p=0.1) + P(X=10\mid p=0.1)$

$= \binom{10}{0}0.1^0 0.9^{10} + \binom{10}{10}0.1^{10}0.9^0 \approx 0.3487$

#2.

a. Simple (分配及參數已指定)

b. Simple (令X為擲die出現的點數，則 $X\in\{1,2,3,4,5,6\}$, $P(X=x)=\frac{1}{6}$)

c. Composite (σ^2未指定)

d. Composite (σ^2未指定)

#3.

a. $H_0: p=0.5$, $H_a: p\neq 0.5$

Under H_0, normal approximation to the binomial distribution

$\mu = np = 100\times 0.5 = 50$

$\sigma = \sqrt{np(1-p)} = \sqrt{100(0.5)(0.5)} = 5$

$\Rightarrow X \overset{d}{\approx} N(50,25)$

$\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true})$

$= P(|X-50|>10 \mid p=0.5) = 1-P(40<x<60\mid p=0.5)$

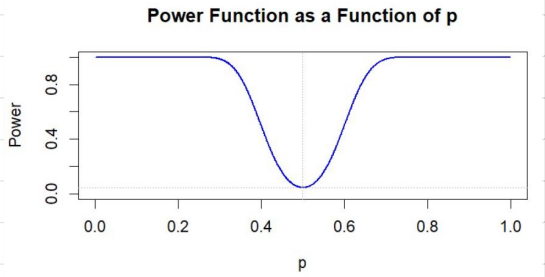
$= 1-P\left(\frac{40-50}{5} < \frac{X-50}{5} < \frac{60-50}{5}\right) = 1-P(-2<Z<2) = 0.0456$ (查表)

where $Z\sim N(0,1)$

b. power = p(reject H_0 | p), where $p \neq 0.5$

Because $X \sim bin(100, p)$, for $0 < p < 1$

compute $p(Z > \frac{60-100p}{\sqrt{100p(1-p)}}, Z < \frac{40-100p}{\sqrt{100p(1-p)}} \mid p)$ where $Z \sim N(0,1)$



#4.

a.

	X	H_0	H_a	λ
x_1		0.2	0.1	2
x_2		0.3	0.4	0.75
x_3		0.3	0.1	3
x_4		0.2	0.4	0.5

reorder from small to large ones!

$\Rightarrow x_4, x_2, x_1, x_3$

b.

	X	H_0	H_1	λ
x_4		0.2	0.4	0.5
x_2		0.3	0.4	0.75
x_1		0.2	0.1	2
x_3		0.3	0.1	3

LR test rejects H_0 for small values of λ

① $\alpha = 0.2$, $p(X = x_4 \mid H_0) = 0.2$

$\Rightarrow$ By NP Lemma, reject H_0 if $X = x_4$

② $\alpha = 0.5$, $p(X = x_4 \mid H_0) + p(X = x_2 \mid H_0) = 0.5$

$\Rightarrow$ By NP Lemma, reject H_0 if $X \in \{x_4, x_2\}$

#5.

a. F, $\alpha = p(\text{reject } H_0 \mid H_0 \text{ is true})$, 而 "$H_0$ is true" 的机率值只可能是 0 或 1.

b. F, α decreasing → 越保護 H_0, RR 越小 → β increasing ⇒ power $= 1-\beta$ decreasing

c. F, $\alpha = p(\text{reject } H_0 \mid H_0 \text{ is true}) \neq p(H_0 \text{ is true} \mid \text{reject } H_0)$

而 "H_0 is true" 的机率值只可能是 0 或 1.

d. F, $p(H_0 \text{ falsely rejected}) = p(\text{data} \in RR \mid H_0 \text{ is true}) = \alpha$

power $= p(\text{data} \in RR \mid H_A \text{ is true}) = 1-\beta$

e. F, Type I error : H_0 is true but data $\in$ RR, 此題陳述少了 "H_0 is true"

f. F 不一定. 常見情況是 Type I error 比 Type II error 嚴重, 但取決於實際情形

ex1、H_0: 沒病 H_A: 有病

Type I error: 健康誤判有病

Type II error: 有病誤判健康

⇒ Type II error 較嚴重.

ex2、H_0: 被告無罪 H_A: 被告有罪

Type I error: 無罪誤判有罪

Type II error: 有罪誤判無罪

⇒ Type I error 較嚴重.

g. F power $= 1-\beta = P(\text{reject } H_0 \mid H_1 \text{ is true})$

，由 test statistic 及 alternative distribution 決定，而非 null distribution.

h. T

7.

$X_1, \cdots, X_n \overset{iid}{\sim} P(\lambda)$

$$\Rightarrow L(\lambda; \underset{\sim}{x}) = f(\underset{\sim}{x}; \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x_i!}$$

For test $H_0: \lambda = \lambda_0$ v.s. $H_1: \lambda = \lambda_1$, $\lambda_1 > \lambda_0$

by N-P lemma the LRT test is

$$\varphi = \begin{cases} 1 & \text{if } \frac{L(\lambda_0)}{L(\lambda_1)} < k \\ r & \text{if } \frac{L(\lambda_0)}{L(\lambda_1)} = k \\ 0 & \text{if } \frac{L(\lambda_0)}{L(\lambda_1)} > k \end{cases}$$

$$\frac{L(\lambda_0)}{L(\lambda_1)} < k$$

$$\Leftrightarrow \frac{e^{-n\lambda_0}\lambda_0^{\sum x_i}}{e^{-n\lambda_1}\lambda_1^{\sum x_i}} = e^{n(\lambda_1-\lambda_0)}\left(\frac{\lambda_0}{\lambda_1}\right)^{\sum x_i} < k$$

$$\Leftrightarrow n(\lambda_1 - \lambda_0) + \sum x_i \ln\left(\frac{\lambda_0}{\lambda_1}\right) < \ln(k)$$

$$\Leftrightarrow \sum x_i > \frac{\ln(k) - n(\lambda_1 - \lambda_0)}{\ln\left(\frac{\lambda_1}{\lambda_0}\right)} = c^* \quad \left(\because \lambda_1 > \lambda_0 \Rightarrow \ln\left(\frac{\lambda_0}{\lambda_1}\right) < 0\right)$$

since $\sum X_i \overset{H_0}{\sim} P(n\lambda_0)$,

$$c^* = \min_c \left(c \mid c \in \mathbb{Z}, \sum_{j=c+1}^{\infty} \frac{(n\lambda_0)^j e^{-n\lambda_0}}{j!} \le \alpha\right)$$

$$\Rightarrow P_{\lambda_0}(\textstyle\sum X_i > c^*) + r P_{\lambda_0}(\sum X_i = c^*) = \alpha$$

$$\Rightarrow r = \frac{\alpha - \sum_{j=c^*+1}^{\infty} \frac{(n\lambda_0)^j e^{-n\lambda_0}}{j!}}{\frac{(n\lambda_0)^{c^*} e^{-n\lambda_0}}{c^*!}}$$

19. (b). $H_0: f_0(x) = \frac{d}{dx} x^2 = 2x$

$H_1: f_1(x) = \frac{d}{dx} x^3 = 3x^2$

$\Lambda(x) = \frac{f_0(x)}{f_1(x)} = \frac{2x}{3x^2} = \frac{2}{3x}$

by N-P Lemma, reject H_0 if $\frac{2}{3x} < k$

$\Leftrightarrow x > \frac{2}{3k} = c^*$

(c). $\alpha = \int_{c^*}^{1} f_0(x)\,dx$

$= \int_{c^*}^{1} 2x\,dx = x^2 \Big|_{c^*}^{1} = 1 - (c^*)^2$

$\Rightarrow c^* = \sqrt{1-\alpha}$

Reject region of level α is $x > \sqrt{1-\alpha}$

(d).

power $= P(X > \sqrt{1-\alpha} \mid H_1 \text{ is true})$

$= \int_{\sqrt{1-\alpha}}^{1} 3x^2\,dx = x^3 \Big|_{\sqrt{1-\alpha}}^{1} = 1 - (1-\alpha)^{\frac{3}{2}}$

29. since g is monotone increasing function

so $T > t_0 \Leftrightarrow g(T) > g(t_0)$

Let $S = g(T)$

$P(S > g(t_0) \mid H_0 \text{ is true}) = P(g(T) > g(t_0) \mid H_0)$

$= P(T > t_0 \mid H_0) = \alpha$

It's a level-α test.

30 (a). 令 T^* 為與 T 相同分佈的 r.v.，以下將 T 固定，則

$$V = P(T^* \geq T \mid H_0)$$
$$= 1 - P(T^* < T \mid H_0)$$
$$= 1 - F(T)$$

(b). if T is continues and c.d.f. is F

then $U = F(T) \sim \mathcal{U}(0,1)$ by prob. integral transform thm.

$$V = 1 - F(T) = 1 - U \sim \mathcal{U}(0,1)$$
$$\Rightarrow V \sim \mathcal{U}(0,1) \text{ under } H_0.$$

(c). 由 (b) $P(V > 0.1 \mid H_0) = \int_{0.1}^{1} 1 \, dx = x\big|_{0.1}^{1} = 0.9$

(d). Sign. level $= P(\text{reg. } H_0 \mid H_0)$

$$= P(V < \alpha \mid H_0)$$

$\because V \sim \mathcal{U}(0,1) \Rightarrow P(V < \alpha) = \alpha$

It is a level-α test.