Problem 1

Given that Y has the probability density function (pdf):

$$f_Y(y) = \begin{cases} \frac{2(\theta-y)}{\theta^2}, & 0 < y < \theta \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $\frac{Y}{\theta}$ is a pivotal quantity.

Let $U = \frac{Y}{\theta}$. To show that U is a pivotal quantity, we need to prove that its distribution does not depend on the parameter θ. We can find the pdf of U using the transformation method. The inverse transformation is $y = \theta u$, and the Jacobian of the transformation is:

$$J = \frac{dy}{du} = \theta$$

The support of Y is $0 < y < \theta$, so the support of U is $0 < \theta u < \theta \implies 0 < u < 1$. The pdf of U, $f_U(u)$, is given by:

$$\begin{aligned} f_U(u) &= f_Y(\theta u) \cdot |J| \\ &= \frac{2(\theta - \theta u)}{\theta^2} \cdot \theta \\ &= \frac{2\theta(1-u)}{\theta^2} \cdot \theta \\ &= 2(1-u) \end{aligned}$$

So, the pdf of U is $f_U(u) = 2(1-u)$ for $0 < u < 1$, and 0 otherwise, which is exactly the pdf of beta(1,2) distribution. Since this pdf is completely independent of the parameter θ, the quantity $U = \frac{Y}{\theta}$ is a pivotal quantity.

(b) Find the cdf of $\frac{Y}{\theta}$.

Let $F_U(u)$ be the cumulative distribution function (cdf) of $U = \frac{Y}{\theta}$. For $0 < u < 1$:

$$\begin{aligned} F_U(u) &= P(U \le u) \\ &= \int_0^u f_U(t)dt \\ &= \int_0^u 2(1-t)dt \\ &= \left[2t - t^2\right]_0^u \\ &= 2u - u^2 \end{aligned}$$

Therefore, the full cdf of $\frac{Y}{\theta}$ is:

$$F_U(u) = \begin{cases} 0, & u \le 0 \\ 2u - u^2, & 0 < u < 1 \\ 1, & u \ge 1 \end{cases}$$

(c) Use (a) to find a 90% confidence interval for θ.

We want to find constants a and b such that:

$$P\left(a \leq \frac{Y}{\theta} \leq b\right) = 0.90$$

Using the equal-tail probability method, we set $P(U \leq a) = 0.05$ and $P(U \geq b) = 0.05$ (which implies $P(U \leq b) = 0.95$). Using the cdf $F_U(u) = 2u - u^2$ derived in part (b):

Find a:

$$2a - a^2 = 0.05$$
$$a^2 - 2a + 0.05 = 0$$

Using the quadratic formula, $a = \frac{2 \pm \sqrt{4-4(1)(0.05)}}{2} = 1 \pm \sqrt{1-0.05} = 1 \pm \sqrt{0.95}$. Since a must be in the interval $(0, 1)$, we choose the negative root:

$$a = 1 - \sqrt{0.95}$$

Find b:

$$2b - b^2 = 0.95$$
$$b^2 - 2b + 0.95 = 0$$

Using the quadratic formula, $b = \frac{2 \pm \sqrt{4-4(1)(0.95)}}{2} = 1 \pm \sqrt{1-0.95} = 1 \pm \sqrt{0.05}$. Since b must be in the interval $(0, 1)$, we choose the negative root:

$$b = 1 - \sqrt{0.05}$$

Now, substitute a and b back into the probability statement:

$$P\left(1 - \sqrt{0.95} \leq \frac{Y}{\theta} \leq 1 - \sqrt{0.05}\right) = 0.90$$

We isolate θ in the inequality:

$$1 - \sqrt{0.95} \leq \frac{Y}{\theta} \implies \theta \leq \frac{Y}{1 - \sqrt{0.95}}$$
$$\frac{Y}{\theta} \leq 1 - \sqrt{0.05} \implies \theta \geq \frac{Y}{1 - \sqrt{0.05}}$$

Combining these, we get:

$$P\left(\frac{Y}{1 - \sqrt{0.05}} \leq \theta \leq \frac{Y}{1 - \sqrt{0.95}}\right) = 0.90$$

Thus, a 90% confidence interval for θ is:

$$\left[\frac{Y}{1 - \sqrt{0.05}}, \frac{Y}{1 - \sqrt{0.95}}\right]$$

Problem 2

Suppose that the random variable Y has a Gamma distribution with parameters $\alpha = 2$ and an unknown rate parameter λ. The probability density function (pdf) of Y is:

$$f_Y(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} = \lambda^2 y e^{-\lambda y}, \quad y > 0$$

(a) Show that $2\lambda Y$ has a χ^2 distribution with 4 degrees of freedom.

Method 1: Moment Generating Function (MGF)
The MGF of a Gamma distribution $Y \sim \text{Gamma}(\alpha = 2, \lambda)$ is:

$$M_Y(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha} = \left(1 - \frac{t}{\lambda}\right)^{-2}, \quad t < \lambda$$

Let $W = 2\lambda Y$. The MGF of W is:

$$\begin{aligned} M_W(t) &= E[e^{tW}] = E[e^{t(2\lambda Y)}] = E[e^{(2\lambda t)Y}] \\ &= M_Y(2\lambda t) \\ &= \left(1 - \frac{2\lambda t}{\lambda}\right)^{-2} \\ &= (1-2t)^{-2} \end{aligned}$$

Recall that the MGF of a Chi-squared distribution with k degrees of freedom (χ^2_k) is $M(t) = (1-2t)^{-k/2}$. By comparing the MGFs, we can see that $(1-2t)^{-2} = (1-2t)^{-4/2}$. Therefore, by the uniqueness theorem of moment generating functions, $W = 2\lambda Y \sim \chi^2_4$.

Method 2: Transformation of Variables (Alternative)
Let $W = 2\lambda Y$. The inverse transformation is $y = \frac{w}{2\lambda}$, and the Jacobian is $J = \frac{dy}{dw} = \frac{1}{2\lambda}$. The support of W is $w > 0$. The pdf of W is:

$$\begin{aligned} f_W(w) &= f_Y\left(\frac{w}{2\lambda}\right) \cdot |J| \\ &= \lambda^2 \left(\frac{w}{2\lambda}\right) e^{-\lambda(\frac{w}{2\lambda})} \cdot \frac{1}{2\lambda} \\ &= \frac{\lambda^2}{4\lambda^2} w e^{-w/2} \\ &= \frac{1}{4} w e^{-w/2} \end{aligned}$$

Which is exactly the pdf of Gamma$(2, \frac{1}{2})$ distribution.The pdf of a χ^2_k distribution is $f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$. i.e. $\chi^2_k \sim Gamma(\frac{K}{2}, \frac{1}{2})$

Substitute $k = 4$:

$$f(x) = \frac{1}{2^{4/2}\Gamma(4/2)} x^{4/2-1} e^{-x/2} = \frac{1}{4 \cdot 1} x^{2-1} e^{-x/2} = \frac{1}{4} x e^{-x/2}$$

Since the pdf of W exactly matches the pdf of a χ^2_4 distribution, $2\lambda Y \sim \chi^2_4$.

(b) Use (a) to derive a 95% confidence interval for λ.

From part (a), we know that $2\lambda Y \sim \chi^2_4$. This is a pivotal quantity since its distribution does not depend on the unknown parameter λ. To find a 95% confidence interval, we use the χ^2_4 distribution to find the critical values $\chi^2_{4,0.975}$ and $\chi^2_{4,0.025}$ such that:

$$P\left(\chi^2_{4,0.975} \leq 2\lambda Y \leq \chi^2_{4,0.025}\right) = 0.95$$

(where $\chi^2_{df,\alpha}$ represents the value such that the right-tail area is α.)

We want to isolate the parameter λ in the middle of the inequality. Since $Y > 0$, we can divide all terms in the inequality by $2Y$ without changing the direction of the inequality signs:

$$\frac{\chi^2_{4,0.975}}{2Y} \leq \lambda \leq \frac{\chi^2_{4,0.025}}{2Y}$$

Therefore, the 95% confidence interval for λ is:

$$\left[\frac{\chi^2_{4,0.975}}{2Y}, \frac{\chi^2_{4,0.025}}{2Y}\right]$$

, where $\chi^2_{4,0.975} = 0.4844$ and $\chi^2_{4,0.025} = 11.1433$

Solution Manual **Homework 8**

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Problem 3

Let $X_1, \ldots, X_n$ be an i.i.d. sample from the Beta distribution with parameters $\beta = 1$ and an unknown α.

(a) Show that $Y_i = -2\alpha \log X_i$ follows exponential distribution $E(1/2)$.

(b) Show that $-2\alpha \sum_{i=1}^n \log X_i$ is a pivotal quantity.

(c) Use (b) to derive a 95% confidence interval for α.

Solution:

Part (a):

The probability density function (pdf) of a Beta distribution with parameters α and $\beta = 1$ is:

$$f_{X_i}(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} x^{\alpha-1}(1-x)^{1-1} = \alpha x^{\alpha-1}, \quad 0 < x < 1$$

We want to find the distribution of $Y_i = -2\alpha \log X_i$. We can use the Cumulative Distribution Function (CDF) method. Note that since $0 < X_i < 1$, we have $0 < Y_i < \infty$.

$$F_{Y_i}(y) = P(Y_i \le y) = P(-2\alpha \log X_i \le y) = P\left(\log X_i \ge -\frac{y}{2\alpha}\right) = P\left(X_i \ge e^{-\frac{y}{2\alpha}}\right)$$

$$F_{Y_i}(y) = \int_{e^{-\frac{y}{2\alpha}}}^1 \alpha x^{\alpha-1} dx = [x^\alpha]_{e^{-\frac{y}{2\alpha}}}^1 = 1 - \left(e^{-\frac{y}{2\alpha}}\right)^\alpha = 1 - e^{-y/2}, \quad y > 0$$

Taking the derivative with respect to y, we get the pdf of Y_i:

$$f_{Y_i}(y) = \frac{d}{dy} F_{Y_i}(y) = \frac{1}{2} e^{-y/2}, \quad y > 0$$

This is precisely the pdf of an exponential distribution with rate parameter 1/2 (or scale parameter 2). Therefore, $Y_i \sim \text{Exp}(1/2)$. Note that this is also equivalent to a chi-squared distribution with 2 degrees of freedom, χ^2_2.

Part (b):

Let $W = -2\alpha \sum_{i=1}^n \log X_i$. We can rewrite this as:

$$W = \sum_{i=1}^n (-2\alpha \log X_i) = \sum_{i=1}^n Y_i$$

From part (a), we know that each Y_i is identically and independently distributed as $\text{Exp}(1/2) \equiv \chi^2_2$. The sum of n independent χ^2_2 random variables follows a chi-squared distribution with $\sum_{i=1}^n 2 = 2n$ degrees of freedom.

$$W \sim \chi^2_{2n}, \text{or equivalently } W \sim \text{Gamma}(n, \frac{1}{2})$$

Because the distribution of W (which is χ^2_{2n}) is completely known and does not depend on the unknown parameter α, the expression $-2\alpha \sum_{i=1}^n \log X_i$ is a pivotal quantity.

Part (c):

To derive a 95% confidence interval for α, we use the pivotal quantity from part (b). Let $\chi^2_{2n,0.975}$ and $\chi^2_{2n,0.025}$ be the critical values of the χ^2_{2n} distribution such that the area to the right is 0.975 and 0.025, respectively.

$$P\left(\chi^2_{2n,0.975} \le -2\alpha \sum_{i=1}^n \log X_i \le \chi^2_{2n,0.025}\right) = 0.95$$

Since $0 < X_i < 1$, the term $\log X_i$ is negative, meaning $-2\sum_{i=1}^{n} \log X_i$ is a positive quantity. We can divide all parts of the inequality by this term without changing the direction of the inequalities:

$$\frac{\chi^2_{2n,0.975}}{-2\sum_{i=1}^{n} \log X_i} \leq \alpha \leq \frac{\chi^2_{2n,0.025}}{-2\sum_{i=1}^{n} \log X_i}$$

Thus, the 95% confidence interval for α is:

$$\left[\frac{\chi^2_{2n,0.975}}{-2\sum_{i=1}^{n} \log X_i}, \frac{\chi^2_{2n,0.025}}{-2\sum_{i=1}^{n} \log X_i}\right]$$

Problem 4 (From textbook, chapter 8, problem 60 (f)(g))
Let $X_1, \ldots, X_n$ be an i.i.d. sample from an exponential distribution with the density function

$$f(x|\tau) = \frac{1}{\tau} e^{-x/\tau}, \quad 0 \le x < \infty$$

(f) Find the form of an approximate confidence interval for τ.
(g) Find the form of an exact confidence interval for τ.

Solution:
Part (f): (註: 此題 (a)-(d) 的答案，請參見 HW4 解答。)

To find an approximate confidence interval, we use the Central Limit Theorem (CLT) and the asymptotic properties of the Maximum Likelihood Estimator (MLE). For the exponential distribution $f(x|\tau) = \frac{1}{\tau} e^{-x/\tau}$, the mean is $E[X] = \tau$ and the variance is $Var(X) = \tau^2$. The MLE of τ is the sample mean, $\hat{\tau} = \bar{X}$. By the Central Limit Theorem, for large n:

$$\sqrt{n}(\bar{X} - \tau) \xrightarrow{d} N(0, \tau^2)$$

To form a confidence interval, we can use Slutsky's theorem and replace the standard deviation τ with its consistent estimator $\bar{X}$:

$$\frac{\sqrt{n}(\bar{X} - \tau)}{\bar{X}} \xrightarrow{d} N(0, 1)$$

For an approximate $(1 - \alpha) \times 100\%$ confidence interval, we use the standard normal critical value $z_{\alpha/2}$:

$$P\left(-z_{\alpha/2} \le \frac{\sqrt{n}(\bar{X} - \tau)}{\bar{X}} \le z_{\alpha/2}\right) \approx 1 - \alpha$$

Isolating τ:

$$-z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}} \le \bar{X} - \tau \le z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}}$$

$$\bar{X} - z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}} \le \tau \le \bar{X} + z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}}$$

Therefore, the form of the approximate confidence interval for τ is:

$$\left[\bar{X} - z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\bar{X}}{\sqrt{n}}\right]$$

Part (g):
To find an exact confidence interval, we must identify a pivotal quantity. Since $X_i \sim \text{Exp}(\frac{1}{\tau})$ (which is a Gamma distribution with $\alpha = 1, \lambda = \frac{1}{\tau}$), scaling the variable yields $\frac{X_i}{\tau} \sim \text{Exp}(1)$. Multiplying by 2, we have $\frac{2X_i}{\tau} \sim \text{Exp}(1/2) \equiv \chi^2_2$. Summing over the n independent observations:

$$\sum_{i=1}^{n} \frac{2X_i}{\tau} = \frac{2\sum_{i=1}^{n} X_i}{\tau} = \frac{2n\bar{X}}{\tau} \sim \chi^2_{2n}, \text{or equivalently } \sum_{i=1}^{n} \frac{2X_i}{\tau} \sim \text{Gamma}(n, \frac{1}{2})$$

This is our pivotal quantity because its distribution (χ^2_{2n}) does not depend on the unknown parameter τ. For an exact $(1 - \alpha) \times 100\%$ confidence interval, let $\chi^2_{2n,1-\alpha/2}$ and $\chi^2_{2n,\alpha/2}$ be the critical values:

$$P\left(\chi^2_{2n,1-\alpha/2} \le \frac{2n\bar{X}}{\tau} \le \chi^2_{2n,\alpha/2}\right) = 1 - \alpha$$

Taking the reciprocal of all terms reverses the direction of the inequalities:

$$\frac{1}{\chi^2_{2n,\alpha/2}} \le \frac{\tau}{2n\bar{X}} \le \frac{1}{\chi^2_{2n,1-\alpha/2}}$$

Multiplying through by $2n\bar{X}$, we obtain the confidence interval bounds for τ:

$$\frac{2n\bar{X}}{\chi^2_{2n,\alpha/2}} \leq \tau \leq \frac{2n\bar{X}}{\chi^2_{2n,1-\alpha/2}}$$

Therefore, the form of the exact confidence interval for τ is:

$$\left[\frac{2n\bar{X}}{\chi^2_{2n,\alpha/2}}, \frac{2n\bar{X}}{\chi^2_{2n,1-\alpha/2}}\right]$$