1. Suppose that Y have the pdf:

$$f_Y(y) = \begin{cases} \frac{2(\theta-y)}{\theta^2}, & 0<y<\theta \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $\frac{Y}{\theta}$ is a pivotal quantity.

(b) Find the cdf of $\frac{Y}{\theta}$.

(c) Use (a) to find a 90% confidence interval for θ.

(a) Let $Q=\frac{Y}{\theta} \Rightarrow Y=\theta Q \Rightarrow J=\frac{dy}{dq}=\theta$, $\theta>0$

$f_Q(q)=f_Y(\theta q)|J|=\frac{2(\theta-\theta q)}{\theta^2}\theta=2(1-q)=\frac{\Gamma(1+2)}{\Gamma(1)\Gamma(2)}q^{1-1}(1-q)^{2-1}$, $q\in(0,1)$

$\Rightarrow Q\sim \text{Beta}(1,2)$

The distribution of Q does not depend on θ.

$\therefore Q=\frac{Y}{\theta}$ is a pivotal quantity.

(b) $F_Q(q)=\int_0^q f_Q(u)\,du=\int_0^q 2(1-u)\,du=2u-u^2\Big|_0^q=2q-q^2$, $q\in(0,1)$

CDF of $\frac{Y}{\theta}=Q$ is $F_Q(q)=\begin{cases} 1, & q\geq 1 \\ 2q-q^2, & q\in(0,1) \\ 0, & \text{otherwise} \end{cases}$

(c) $0.9=P(a\leq Q\leq b)=P(a\leq\frac{Y}{\theta}\leq b)=P(\frac{Y}{b}\leq\theta\leq\frac{Y}{a})$

$\left(\frac{Y}{b},\frac{Y}{a}\right)$ is a 90% confidence interval for θ, where a,b satisfy

$0.9=F_Q(b)-F_Q(a)=2b-b^2-2a+a^2=a^2-b^2-2(a-b)$

Find a set of solutions for a and b.

Let $F_Q(a)=0.05$, $a\in(0,1)$

$\Rightarrow 2a-a^2=0.05 \Rightarrow a=0.0254$ or 1.9746 (不合 $\because 1.9746\notin(0,1)$)

Then $F_Q(b)=0.9+F_Q(a)=0.9+0.05$, $b\in(0,1)$

$\Rightarrow 2b-b^2=0.95 \Rightarrow b=0.7764$ or 1.2236 (不合 $\because 1.2236\notin(0,1)$)

$\left(\frac{Y}{0.7764},\frac{Y}{0.0254}\right)$ is a 90% confidence interval for θ.

2. Suppose that the random variable Y has a Gamma distribution with parameters $\alpha = 2$ and an unknown λ.

   (a) Show that $2\lambda Y$ has a χ^2 distribution with 4 degrees of freedom.

   (b) Use (a) to derive a 95% confidence interval for λ.

   [Hint: Lecture Notes, Ch1-6, p.74, item 6 and p.77, item 2.]

(a) $X = 2\lambda Y \Rightarrow Y = \frac{X}{2\lambda} \Rightarrow J = \frac{1}{2\lambda}$, $\lambda > 0$

$$f_X(x) = f_Y\left(\frac{x}{2\lambda}\right)|J| = \frac{\lambda^2 \left(\frac{x}{2\lambda}\right)^{2-1} e^{-\lambda\left(\frac{x}{2\lambda}\right)}}{\Gamma(2)} \frac{1}{2\lambda} = \frac{x e^{-\frac{x}{2}}}{4} = \frac{\left(\frac{1}{2}\right)^2 x^{2-1} e^{-\frac{1}{2}x}}{\Gamma(2)}, \quad x \in (0, \infty)$$

$$\Rightarrow X = 2\lambda Y \sim \Gamma\left(2, \lambda = \tfrac{1}{2}\right) = \chi^2(4)$$

(b) $$0.95 = P\left(\chi^2_{0.025}(4) \le X \le \chi^2_{0.975}(4)\right)$$

$$= P\left(\chi^2_{0.025}(4) \le 2\lambda Y \le \chi^2_{0.975}(4)\right)$$

$$= P\left(\frac{\chi^2_{0.025}(4)}{2Y} \le \lambda \le \frac{\chi^2_{0.975}(4)}{2Y}\right)$$

$\left(\frac{\chi^2_{0.025}(4)}{2Y}, \frac{\chi^2_{0.975}(4)}{2Y}\right)$ is a 95% confidence interval for λ.

3. Let $X_1, \ldots, X_n$ be an i.i.d. sample from the Beta distribution with parameters $\beta = 1$ and an unknown α.

(a) Show that $Y_i = -2\alpha \log X_i$ follows exponential distribution $E(\frac{1}{2})$.

(b) Show that $-2\alpha \sum_{i=1}^{n} \log X_i$ is a pivotal quantity.

(c) Use (b) to derive a 95% confidence interval for α.

pf.

$f_X(x) = \alpha x^{\alpha-1}, \quad 0 < x < 1$

(a) Define $Z_i = -\ln X_i$. $\forall z > 0$

$$P(Z_i \geq z) = P(X_i \leq e^{-z}) = \int_0^{e^{-z}} \alpha x^{\alpha-1} dx = e^{-z\alpha}$$

$Z_i \sim E(\alpha)$. Since $Y_i = 2\alpha Z_i$, a constant multiple of an exponential variable with

$$Y_i \sim E\left(\frac{\alpha}{2\alpha}\right) = E\left(\frac{1}{2}\right)$$

(b) Since $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} E(\frac{1}{2})$

$S_n = \sum Y_i = -2\alpha \sum \ln X_i \sim \Gamma(n, \frac{1}{2}) = \chi^2_{2n}$

Since the dist. of S_n does not contain α

$\Rightarrow$ S_n is pivotal quantity.

(c) let

$W = -\sum \ln X_i > 0 \qquad S_n = 2\alpha W$

Since $S_n \sim \chi^2_{2n}$

$$P(\chi^2_{2n,0.025} \leq S_n \leq \chi^2_{2n,0.975}) = 0.95.$$

we have

$$\frac{\chi^2_{2n,0.025}}{-2\sum \ln X_i} \leq \alpha \leq \frac{\chi^2_{2n,0.975}}{-2\sum \ln X_i} \quad \text{with prob. } 0.95$$

Therefore, the C.I. for α is

$$\left(\frac{\chi^2_{2n,0.025}}{-2\sum \ln X_i}, \frac{\chi^2_{2n,0.975}}{-2\sum \ln X_i} \right)$$

證

60. Let $X_1, \ldots, X_n$ be an i.i.d. sample from an exponential distribution with the density function

$$f(x|\tau) = \frac{1}{\tau}e^{-x/\tau}, \qquad 0 \le x < \infty$$

a. Find the mle of τ.
b. What is the exact sampling distribution of the mle?
c. Use the central limit theorem to find a normal approximation to the sampling distribution.
d. Show that the mle is unbiased, and find its exact variance. (*Hint:* The sum of the X_i follows a gamma distribution.)
e. Is there any other unbiased estimate with smaller variance?
f. Find the form of an approximate confidence interval for τ.
g. Find the form of an exact confidence interval for τ.

p.f. (f) (g)

By Hw 4, we have MLE is $\hat{\tau} = \bar{X} = \frac{1}{n}\sum X_i$

(f)

Because

$$\sqrt{n}\,\frac{\bar{X}-\tau}{\tau} \sim N(0,1),$$

we approx. have

$$P\left(-z_{\frac{\alpha}{2}} \le \sqrt{n}\,\frac{\bar{X}-\tau}{\tau} \le z_{\frac{\alpha}{2}}\right) \approx 1-\alpha$$

a $100(1-\alpha)\%$ approximate C.I. of τ is

$$\left(\frac{\bar{X}}{1+z_{\frac{\alpha}{2}}/\sqrt{n}}, \frac{\bar{X}}{1-z_{\frac{\alpha}{2}}/\sqrt{n}}\right)$$

Another method. (CH8, p83, 84)

By Thm. 6.22, MLE $\hat{\tau}$ satisfies

$$\sqrt{nI(\tau)}\,(\hat{\tau}-\tau) \sim N(0,1),$$ which give

asymptotic C.I $\hat{\tau} \pm z_{\frac{\alpha}{2}} \frac{1}{\sqrt{nI(\hat{\tau})}}$

Fisher Info.

$$\ell(\tau) = \ln f(x|\tau) = -\ln(\tau) - \frac{x}{\tau}$$

$$\frac{\partial \ell(\tau)}{\partial \tau} = \frac{-1}{\tau} + \frac{x}{\tau^2} \qquad \frac{\partial^2 \ell(\tau)}{\partial \tau^2} = \frac{1}{\tau^2} - \frac{2x}{\tau^3}$$

$$I(\tau) = -E\left[\frac{\partial^2 \ell(\tau)}{\partial \tau^2}\right] = \frac{1}{\tau^2} \qquad (E[X] = \tau)$$

Hence we have $100(1-\alpha)\%$ approx. C.I.

$$\bar{X} \pm z_{\frac{\alpha}{2}} \frac{\bar{X}}{\sqrt{n}} \qquad (\hat{\tau} = \bar{X})$$

$$\Leftrightarrow \left(\bar{X} - z_{\frac{\alpha}{2}}\frac{\bar{X}}{\sqrt{n}},\ \bar{X} + z_{\frac{\alpha}{2}}\frac{\bar{X}}{\sqrt{n}}\right)$$

(g) let $S = \sum X_i$. Since $Exp(\tau) = \Gamma(1, \frac{1}{\tau})$

$\frac{2S}{\tau} \sim \chi^2_{2n} = \Gamma(n, \frac{1}{\frac{2}{\tau}\cdot\tau}) = \chi^2_{2n}$ is a pivotal quantity

$$\Rightarrow P\left(\chi^2_{2n,1-\frac{\alpha}{2}} \le \frac{2S}{\tau} \le \chi^2_{2n,\frac{\alpha}{2}}\right) = 1-\alpha$$

$\Rightarrow$ a $100(1-\alpha)\%$ C.I. of τ is $\left(\frac{2n\bar{X}}{\chi^2_{2n,1-\frac{\alpha}{2}}},\ \frac{2n\bar{X}}{\chi^2_{2n,\frac{\alpha}{2}}}\right)$

(exact $1-\alpha$ C.I.)