(i) Ch8 #57

(a) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$E(\frac{(n-1)S^2}{\sigma^2}) = n-1 \Rightarrow E(S^2) = E(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2) = \sigma^2$

$\Rightarrow S^2$ is an unbiased estimator of σ^2

$E(\hat{\sigma}^2) = E(\frac{n-1}{n}S^2) = \frac{n-1}{n}E(S^2) = \frac{n-1}{n}\sigma^2$

$\Rightarrow \hat{\sigma}^2$ is not an unbiased estimator of σ^2

(b) $Var(\frac{(n-1)S^2}{\sigma^2}) = 2(n-1) \Rightarrow Var(S^2) = (\frac{\sigma^2}{n-1})^2 \cdot 2(n-1) = \frac{2\sigma^4}{n-1}$, $bias(S^2) = 0$.

$MSE(S^2) = \frac{2\sigma^4}{n-1} + 0^2 = \frac{2\sigma^4}{n-1}$

$Var(\hat{\sigma}^2) = Var(\frac{n-1}{n}S^2) = (\frac{n-1}{n})^2 \cdot \frac{2\sigma^4}{n-1} = \frac{(n-1)2\sigma^4}{n^2}$, $bias(\hat{\sigma}^2) = \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{-1}{n}\sigma^2$

$MSE(\hat{\sigma}^2) = \frac{2(n-1)\sigma^4}{n^2} + (\frac{-1}{n}\sigma^2)^2 = \frac{(2n-2)\sigma^4+\sigma^4}{n^2} = \frac{(2n-1)\sigma^4}{n^2}$

$MSE(S^2) - MSE(\hat{\sigma}^2) = \frac{2\sigma^4}{n-1} - \frac{(2n-1)\sigma^4}{n^2} = \frac{2n^2\sigma^4 - (2n^2-3n+1)\sigma^4}{n^2(n-1)} = \frac{(3n-1)\sigma^4}{n^2(n-1)}$

since $n>1 \Rightarrow MSE(S^2) - MSE(\hat{\sigma}^2) > 0 \Rightarrow MSE(S^2) > MSE(\hat{\sigma}^2)$

(c) $\sum_{i=1}^{n}(X_i-\bar{X})^2 = (n-1)S^2$

$\Rightarrow \rho\sum_{i=1}^{n}(X_i-\bar{X})^2 = \rho(n-1)S^2$

$E(\rho\sum_{i=1}^{n}(X_i-\bar{X})^2) = \rho(n-1)\sigma^2$, $bias(\rho\sum_{i=1}^{n}(X_i-\bar{X})^2) = [\rho(n-1)-1]\sigma^2$

$Var(\rho\sum_{i=1}^{n}(X_i-\bar{X})^2) = Var(\rho(n-1)S^2) = \rho^2(n-1)^2 \cdot \frac{2\sigma^4}{n-1} = 2\rho^2(n-1)\sigma^4$

$MSE(\rho\sum_{i=1}^{n}(X_i-\bar{X})^2) = 2\rho^2(n-1)\sigma^4 + [(\rho(n-1)-1)\sigma^2]^2$

$= 2\rho^2(n-1)\sigma^4 + [(\rho(n-1))^2 - 2\rho(n-1)+1]\sigma^4 = ((n^2-1)\rho^2 - 2(n-1)\rho + 1)\sigma^4$

$\frac{\partial}{\partial\rho}MSE(\rho\sum_{i=1}^{n}(X_i-\bar{X})^2) = \frac{\partial}{\partial\rho}((n^2-1)\rho^2 - 2(n-1)\rho+1)\sigma^4 = [2(n^2-1)\rho - 2(n-1)]\sigma^4$

$\underline{\text{set}}\ 0$

$\Rightarrow \rho = \frac{1}{n+1}$

$\frac{\partial^2}{\partial\rho^2}MSE(\rho\sum_{i=1}^{n}(X_i-\bar{X})^2)\Big|_{\rho=\frac{1}{n+1}} = 2(n^2-1)\sigma^4 > 0$.

$\Rightarrow$ When $\rho = \frac{1}{n+1}$, $\rho\sum_{i=1}^{n}(X_i-\bar{X})^2$ has the minimal MSE.

(ii) Ch8 #60

(a)-(d)的答案亦可參見HW4的解答

(e) $f(x|\tau)=\frac{1}{\tau}e^{-\frac{x}{\tau}} \quad 0\le x<\infty$

$=0 \quad$ O.W

$L(\tau|\underset{\sim}{x})=\prod_{i=1}^{n} f(x_i|\tau)=\tau^{-n}e^{-\frac{\sum x_i}{\tau}}$

$\ell(\tau|\underset{\sim}{x})=-n\log\tau-\frac{\sum_{i=1}^{n}x_i}{\tau}$

$\frac{\partial \ell(\tau|\underset{\sim}{x})}{\partial\tau}=\frac{-n}{\tau}+\frac{\sum_{i=1}^{n}x_i}{\tau^2}\overset{\text{set}}{=}0\Rightarrow \hat{\tau}_{MLE}=\bar{X}$ HW4

$\left(\frac{\partial^2\ell(\tau|\underset{\sim}{x})}{\partial\tau^2}\Big|_{\tau=\bar{x}}=\frac{n}{\tau^2}-\frac{2\sum_{i=1}^{n}x_i}{\tau^3}\Big|_{\tau=\bar{x}}<0\right)$

The Fisher information of one observation is

$I(\tau)=-E\left(\frac{\partial^2}{\partial\tau^2}\log f(x|\tau)\right)=-E\left(\frac{\partial^2}{\partial\tau^2}\left(\log\tau-\frac{x}{\tau}\right)\right)=-E\left(\frac{1}{\tau^2}-\frac{2x}{\tau^3}\right)=-\frac{1}{\tau^2}+\frac{2}{\tau^2}=\frac{1}{\tau^2}$

⇒In regular condition, for any unbiased estimator $\tilde{\tau}$ of τ

$Var(\tilde{\tau})\ge\frac{1}{nI(\tau)}=\frac{\tau^2}{n}=CRLB(\tau)$

$E(\bar{X})=\tau,\ Var(\bar{X})=\frac{\tau^2}{n}$ HW4

⇒$\hat{\tau}_{MLE}=\bar{X}$ is an unbiased estimator of τ and $\hat{\tau}_{MLE}=\bar{X}$ achieve the CRLB(τ) HW4

⇒No other unbiased estimator can have a smaller variance than MLE

(iii) $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta, 1)$

(a)

$$L(\theta|\underset{\sim}{X}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\theta)^2}{2}}$$

$$= (2\pi)^{-\frac{n}{2}} e^{-\sum_{i=1}^{n} \frac{(x_i-\theta)^2}{2}}$$

$$= (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}(\sum_{i=1}^{n} x_i^2 - 2n\bar{X}\theta + n\theta^2)}$$

let $T(X) = \bar{X}$ $C(\theta) = n\theta$. (1-parameter exponential family)

$\bar{X}$ is a C.S.S (complete and sufficient statistic) of θ

$$E(\bar{X}^2) = Var(\bar{X}) + [E(\bar{X})]^2$$

$$= \frac{1}{n} + \theta^2$$

$$\therefore E(\bar{X}^2 - \frac{1}{n}) = \theta^2$$

by Rao-blackwell and Lehmann-Scheffe's thm

$\bar{X}^2 - \frac{1}{n}$ is the UMVUE of θ^2

(b) $Var(\bar{X}^2 - \frac{1}{n}) = Var(\bar{X}^2) = E(\bar{X}^4) - [E(\bar{X}^2)]^2$

$$M_{\bar{X}}(t) = e^{\theta t + \frac{t^2}{2n}}$$

$$E(\bar{X}^4) = \frac{\partial^4 M_{\bar{X}}(t)}{\partial t^4}\Big|_{t=0} = e^{\theta t + \frac{t^2}{2n}} \left[(\theta + \frac{t}{n})^4 + \frac{6}{n}(\theta + \frac{t}{n})^2 + \frac{3}{n^2} \right] \Big|_{t=0}$$

$$= \theta^4 + \frac{6}{n}\theta^2 + \frac{3}{n^2}$$

$$[E(\bar{X}^2)]^2 = (\frac{1}{n} + \theta^2)^2 = \frac{1}{n^2} + \frac{2\theta^2}{n} + \theta^4$$

$$\therefore Var(\bar{X}^2 - \frac{1}{n}) = \cancel{\theta^4} + \frac{6}{n}\theta^2 + \frac{3}{n^2} - \cancel{\theta^4} - \frac{2\theta^2}{n} - \frac{1}{n^2} = \frac{4}{n}\theta^2 + \frac{2}{n^2}$$

$$\ln f(x|\theta) = -\frac{1}{2}\log 2\pi - \frac{1}{2}(x-\theta)^2$$

$$\frac{\partial}{\partial\theta}\ln f(x|\theta) = x-\theta$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(x|\theta) = -1$$

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x|\theta)\right] = 1$$

$$\Rightarrow CRlb = \frac{[g'(\theta)]^2}{nI(\theta)} = \frac{4\theta^2}{n}$$

and $\quad Var(\bar{X}^2 - \frac{1}{n}) = \frac{4}{n}\theta^2 + \frac{2}{n^2} > \frac{4\theta^2}{n}$

$\therefore$ CRlb isn't attained for the UMVUE

(iv) $Y_1, \dots, Y_n \overset{iid}{\sim} Ber(p)$

(a)
$$f(y|p) = p^y(1-p)^{1-y}$$
$$= \left(\frac{p}{1-p}\right)^y 1-p$$
$$= e^{y\log\frac{p}{1-p}}\cdot(1-p)$$

$T(y) = y \quad c(p) = \log\frac{p}{1-p}$ (exponential family)

For $Y_1, \dots, Y_n \quad W = \sum_{i=1}^n Y_i$ is a C.S.S of p

(b) $E(T) = P(Y_1=1, Y_2=0) = P(Y_1=1)P(Y_2=0) = p(1-p)$

(c) $P(T=1|W=w) = \dfrac{P(Y_1=1, Y_2=0, \sum_{i=1}^{n} Y_i = w)}{P(\sum_{i=1}^{n} Y_i = w)} = \dfrac{P(Y_1=1)P(Y_2=0)P(\sum_{i=3}^{n} Y_i = w-1)}{P(\sum_{i=1}^{n} Y_i = w)}$

$\sum_{i=1}^{n} Y_i \sim Bin(n,p)$

$\sum_{i=3}^{n} Y_i \sim Bin(n-2,p)$

$= \dfrac{p(1-p)\binom{n-2}{w-1}p^{w-1}(1-p)^{n-w-1}}{\binom{n}{w}p^{w}(1-p)^{n-w}} = \dfrac{w(n-w)}{n(n-1)}$

(d) $E(T|W=w) = 1 \cdot P(T=1|W=w) = \dfrac{w(n-w)}{n(n-1)}$

$= \dfrac{w(n-w)}{n^2}\dfrac{n}{n-1}$

$= \dfrac{n}{n-1}\left[\dfrac{w}{n}\left(1-\dfrac{w}{n}\right)\right]$

$E(T|W) = \dfrac{n}{n-1}\left[\dfrac{w}{n}\left(1-\dfrac{w}{n}\right)\right] = \dfrac{n}{n-1}\left[\bar{Y}(1-\bar{Y})\right]$

$\because$ T is unbiased of $p(1-p)$ and W is a C.S.S of p

by Rao-Blackwell and Lehmann Scheffe's thm $E(T|W)$ is the UMVUE of $p(1-p)$

(V) $Y_1, \dots, Y_n \overset{iid}{\sim} f(y|\theta) = \dfrac{3y^2}{\theta^3}, \ 0 \le y \le \theta$

(a) $\because$ the support of $f(y|\theta)$ depends on θ $\therefore$ the pdf is not exponential family

(b) $f(y_1 \cdots y_n|\theta) = \prod_{i=1}^{n} \dfrac{3y_i^2}{\theta^3} I_{[0,\theta]}(y_i)$

$= \dfrac{3^n \prod_{i=1}^{n} y_i^2}{\theta^{3n}} I_{[0,\theta]}(y_{(n)})$ by factorization thm

$Y_{(n)}$ is a S.S of θ

(c) $f_{Y_{(n)}}(y|\theta) = n[F(y)]^{n-1} f(y)$

$F(y) = \int_0^y \frac{3t^2}{\theta^3}\,dt = \frac{1}{\theta^3}[t^3]_0^y = \frac{y^3}{\theta^3}$

$\therefore f_{Y_{(n)}}(y|\theta) = n\left(\frac{y^3}{\theta^3}\right)^{n-1} \cdot \frac{3y^2}{\theta^3} = \begin{cases} \frac{3ny^{3n-1}}{\theta^{3n}} & 0 \le y \le \theta \\ 0 & \text{o.w.} \end{cases}$

(d) $T = Y_{(n)} \in (0, \infty)$

Let $E[g(t)] = 0 \quad \forall\, \theta > 0$

$\int_0^\theta g(t) \frac{3nt^{3n-1}}{\theta^{3n}}\,dt = 0 \quad \forall\, \theta > 0$

$\Rightarrow \int_0^\theta g(t)\, t^{3n-1}\,dt = 0 \quad \forall\, \theta > 0$

by F.T.O.C.

$\frac{d}{d\theta}\int_0^\theta g(t)t^{3n-1}\,dt = \frac{d}{d\theta}0 \quad \forall\, \theta > 0$

$g(\theta)\theta^{3\theta-1} = 0 \quad \forall\, \theta > 0$

$\because \theta > 0 \quad \therefore g(\theta) = 0$

Hence $Y_{(n)}$ is complete

(e) $E(T) = \int_0^\theta t \frac{3nt^{3n-1}}{\theta^{3n}}\,dt = \frac{3n}{\theta^{3n}}\int_0^\theta t^{3n}\,dt = \frac{3n}{\theta^{3n}}\left[\frac{1}{3n+1}t^{3n+1}\right]_0^\theta$

$= \frac{3n}{3n+1}\theta$

$\Rightarrow E\left(\frac{3n+1}{3n}T\right) = \theta$ $\quad \frac{3n+1}{3n}Y_{(n)}$ is unbiased for θ and $Y_{(n)}$ is c.s.s of θ by Rao-blackwell and Lehmann-Scheffe's thm $\frac{3n+1}{3n}Y_{(n)}$ is a UMVUE of θ