MATH 2820, 2025, Homework 7

(1) Let $X_1, \ldots, X_n$ be an i.i.d. sample from $N(\theta, 1)$.

(a) Show that $\overline{X}^2 - \frac{1}{n}$ is a UMVUE of $g(\theta) = \theta^2$.

(b) Examine whether the Cramer-Rao bound is attained for the UMVUE.

(2) Let $Y_1, \ldots, Y_n$ be an i.i.d. sample from Bernoulli distribution $B(p)$. Find a UMVUE of $p(1-p)$, which is a term in the variance of Y_i or $W = \sum_{i=1}^{n} Y_i$, by using the following steps:

(a) Show that W is a sufficient and complete statistic.

(b) Let

$$T = \begin{cases} 1, & \text{if } Y_1{=}1 \text{ and } Y_2{=}0 \\ 0, & \text{otherwise} \end{cases}$$

Show that $\mathrm{E}(T) = p(1-p)$.

(c) Show that

$$P(T = 1 | W = w) = \frac{w(n-w)}{n(n-1).}$$

(d) Show that

$$\mathrm{E}(T|W) = \frac{n}{n-1}\left[\frac{W}{n}\left(1 - \frac{W}{n}\right)\right] = \frac{n}{n-1}\overline{Y}(1-\overline{Y})$$

and, explain why $\frac{n\overline{Y}(1-\overline{Y})}{(n-1)}$ is a UMVUE of $p(1-p)$.

(3) Let $Y_1, \ldots, Y_n$ be an i.i.d. sample from the pdf:

$$f(y|\theta) = \begin{cases} \frac{3y^2}{\theta^3}, & \text{if } 0 \le y \le \theta \\ 0, & \text{otherwise} \end{cases}$$

where $0 < \theta < \infty$.

(a) Check whether the pdfs form an exponential family.

(b) Show that $Y_{(n)} = \max\{Y_1, \ldots, Y_n\}$ is a sufficient statistic.

(c) Show that $Y_{(n)}$ has pdf:

$$f_{Y_{(n)}}(y|\theta) = \begin{cases} \frac{3ny^{3n-1}}{\theta^{3n}}, & \text{if } 0 \le y \le \theta \\ 0, & \text{otherwise} \end{cases}$$

(d) Show that $Y_{(n)}$ is complete by definition.

(e) Find a UMVUE of θ.