Chapter 8

16(d)

Density:

$$f(x|\sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right)$$

Likelihood:

$$L(\sigma; x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{2\sigma} \exp\left(-\frac{|x_i|}{\sigma}\right) = \left(\frac{1}{2\sigma}\right)^n \exp\left(-\frac{1}{\sigma}\sum_{i=1}^{n} |x_i|\right)$$

Using the Neyman-Fisher factorization theorem:

$$L(\sigma; \mathbf{x}) = h(\mathbf{x}) \cdot g\left(T(\mathbf{x}), \sigma\right)$$

where:

$T(\mathbf{x}) = \sum_{i=1}^{n} |x_i|$; $h(\mathbf{x}) = 1$; $g(T, \sigma) = \left(\frac{1}{2\sigma}\right)^n \exp\left(-\frac{T}{\sigma}\right)$

Hence, $\sum_{i=1}^{n} |X_i|$ is a sufficient statistic for σ.

Show that the pdfs form an exponential family and find a sufficient and complete statistic.

We rewrite the density function as:

$$f(x|\sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right) = \exp\left(-\log(2\sigma) - \frac{|x|}{\sigma}\right)$$

Let $\eta = -\frac{1}{\sigma}$, then:

$$f(x|\eta) = \exp\left(\eta|x| + \log(-\eta) - \log 2\right)$$

This is the canonical form of a one-parameter exponential family with:

- Sufficient statistic $T(x) = |x|$
- Natural parameter $\eta = -1/\sigma < 0$

Since this is a one-parameter exponential family with open natural parameter space, the sufficient statistic $\sum_{i=1}^{n} |X_i|$ is also complete.

18(d)

Density:

$$f(x|\alpha) = \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x^{\alpha-1}(1-x)^{2\alpha-1}, \quad x \in [0, 1]$$

Likelihood:

$$L(\alpha;\mathbf{x}) = \left(\frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)}\right)^n \prod_{i=1}^n x_i^{\alpha-1}(1-x_i)^{2\alpha-1}$$
$$= \left(\frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)}\right)^n \exp\{(\alpha-1)\sum_{i=1}^n \ln x_i + (2\alpha-1)\sum_{i=1}^n \ln(1-x_i)\}$$
$$= \left(\frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)}\right)^n \exp\{\alpha(\sum_{i=1}^n \ln x_i + 2\sum_{i=1}^n \ln(1-x_i)) - (\sum_{i=1}^n \ln x_i + \sum_{i=1}^n \ln(1-x_i))\}$$

Using the Neyman-Fisher factorization theorem:

$$L(\sigma;\mathbf{x}) = h(\mathbf{x}) \cdot g\left(T(\mathbf{x}), \sigma\right)$$

where:

$$T(\mathbf{x}) = (\sum_{i=1}^n \ln x_i + 2\sum_{i=1}^n \ln(1-x_i))$$

Hence, $(\sum_{i=1}^n \ln X_i + 2\sum_{i=1}^n \ln(1-X_i))$ is a sufficient statistic for σ.

Show that the pdfs form an exponential family and find a sufficient and complete statistic

We can write the density function as:

$$f(x|\alpha) = \exp\left(\ln\frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} + \alpha\ln(x(1-x)^2) - \ln(x(1-x))\right)$$

This confirms that the family forms an exponential family with:

- Natural parameter: α
- Sufficient statistics: $\sum_{i=1}^n \ln x_i + 2\sum_{i=1}^n \ln(1-x_i)$

Since this is a one-parameter exponential family with natural parameter $\alpha \in (0,\infty)$ (an open interval), the sufficient statistic $\sum_{i=1}^n \ln X_i + 2\sum_{i=1}^n \ln(1-X_i)$ is also complete by standard results on exponential families.

21(c)

Density:

$$f(x|\theta) = e^{-(x-\theta)}\mathbb{I}(x \geq \theta)$$

Likelihood:

$$L(\theta;\mathbf{x}) = e^{-\sum x_i + n\theta}\mathbb{I}(\theta \leq X_{(1)})$$

This can be written as:

$$L(\theta;\mathbf{x}) = h(\mathbf{x}) \cdot g(T(\mathbf{x}),\theta)$$

where $T(\mathbf{x}) = X_{(1)} = \min\{X_1,\ldots,X_n\}$. Thus,

$$X_{(1)} \text{ is a sufficient statistic for } \theta.$$

Show that $X_{(1)}$ is complete by definition and examine whether the pdfs form an exponential family

The density function is:

$$f(x|\theta) = e^{-(x-\theta)} \cdot \mathbb{I}(x \geq \theta) = \exp(-x+\theta) \cdot \mathbb{I}(x \geq \theta)$$

The support depends on the parameter θ, which violates a key condition of exponential families. Therefore,

$$f(x|\theta) \text{ does not form an exponential family}$$

Then, prove $X_{(1)}$ is complete, consider its distribution:

$$f_{X_{(1)}}(x) = ne^{-n(x-\theta)} \cdot \mathbb{I}(x \geq \theta)$$

Suppose:

$$\mathbb{E}\left[g(X_{(1)})\right] = 0 \quad \forall \theta$$

Then:

$$\int_0^\infty g(u) \cdot ne^{-n(u-\theta)} du = 0$$

Differentiate both sides with respect to θ:

$$\frac{d}{d\theta} \int_0^\infty g(u) ne^{-n(u-\theta)} du = \frac{d}{d\theta} 0 = 0$$

By chain rule:

$$-g(\theta) ne^0 = 0 \Rightarrow g(\theta) = 0 \quad \forall \theta$$

Therefore:

$$X_{(1)} \text{ is a complete statistic of } \theta.$$

Furthermore, the original density is:

$$f(x|\theta) = e^{-(x-\theta)}, \quad x > \theta$$

Since the range of X depends on the parameter θ, we conclude:

$$\therefore f(x|\theta) \text{ does not belong to an exponential family.}$$

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Consider a muon decay setting where the original record is $x = \cos\theta \in [-1, 1]$, with density:

$$f(x|\alpha) = \frac{1}{2}(1 + \alpha x), \quad |\alpha| \leq 1$$

Instead of observing x, we only record whether $x > 0$ (forward) or $x < 0$ (backward).

(a)

Define an indicator variable:

$$Y_i = \begin{cases} 1, & \text{if } x_i > 0 \quad \text{(forward)} \\ 0, & \text{if } x_i < 0 \quad \text{(backward)} \end{cases}$$

Then we observe i.i.d. binary data: $Y_i \sim \text{Bernoulli}(p)$. The maximum likelihood estimator (MLE) of p is $\hat{p} = \bar{Y} = \frac{1}{n}\sum Y_i$. Because

$$L(p) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i}$$

$$l(p) = \sum_{i=1}^{n} y_i \log p + (1-y_i)\log(1-p)$$

$$\frac{\partial}{\partial p} l(p) = \frac{\sum_{i=1}^{n} y_i}{p} + \frac{(n - \sum_{i=1}^{n} y_i)}{(1-p)}$$

Let the derivative be 0:

$$\frac{\sum_{i=1}^{n} y_i}{p} = \frac{(n - \sum_{i=1}^{n} y_i)}{(1-p)}$$

$$\Rightarrow \hat{p} = \bar{Y} = \frac{1}{n}\sum Y_i, \frac{\partial^2}{\partial p^2} l(p) < 0$$

Calculate $p = P(x > 0)$:

$$p(\alpha) = \int_0^1 \frac{1}{2}(1+\alpha x)dx = \frac{1}{2}\left[1 + \frac{\alpha}{2}\right] = \frac{1}{2} + \frac{\alpha}{4} = \mu_1$$

Using the sample mean:

$$\hat{\mu_1} = \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \Rightarrow \hat{\alpha} = 4\bar{Y} - 2$$

is the moment estimator based on Y_i's.

Since MLE of p is $\hat{p} = \bar{Y}$,then this is the maximum likelihood estimator (MLE) based on Y_i's by invariance property of MLE.

(b)

Since $Y_i \sim \text{Bernoulli}\left(\frac{1}{2} + \frac{\alpha}{4}\right)$ Hence,

$$\text{Var}(Y_i) = \left(\frac{1}{2} + \frac{\alpha}{4}\right)\left(1 - \frac{1}{2} - \frac{\alpha}{4}\right) = \frac{1}{4} - \frac{\alpha^2}{16}$$

Then

$$\text{Var}(\bar{Y}) = \frac{1}{n}\left(\frac{1}{4} - \frac{\alpha^2}{16}\right)$$

Thus,

$$\text{Var}(\hat{\alpha}) = 16 \cdot \text{Var}(\bar{Y}) = \frac{4 - \alpha^2}{n}$$

Next, Method of Moments Estimator based on X_i's: Based on $\mathbb{E}[X] = \frac{\alpha}{3}$, we define:

$$\bar{\alpha} = 3\bar{X}$$

Therefore

$$\mathbb{E}[X^2] = \int_{-1}^{1} x^2 \cdot \frac{1}{2}(1+\alpha x)dx = \frac{1}{3}$$
$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{3} - \left(\frac{\alpha}{3}\right)^2 = \frac{1-\alpha^2/3}{3}$$
$$\text{Var}(\bar{\alpha}) = 9 \cdot \text{Var}(\bar{X}) = \frac{3-\alpha^2}{n}$$

Then, Maximum Likelihood Estimator based on X_i's:we define $\widetilde{\alpha}$, although $\widetilde{\alpha}$ exist, it does not admit a closed-form expression.

Log-likelihood:

$$\log f(x|\alpha) = \log(1+\alpha x) - \log 2 \Rightarrow \ell''(\alpha) = -\sum \frac{x^2}{(1+\alpha x)^2}$$

Fisher Information:

$$I(\alpha) = \mathbb{E}\left[\frac{x^2}{(1+\alpha x)^2}\right] = \int_{-1}^{1} \frac{x^2}{(1+\alpha x)^2} \cdot \frac{1}{2}(1+\alpha x)dx = \int_{-1}^{1} \frac{x^2}{2(1+\alpha x)}dx$$

This has a closed form:

$$I(\alpha) = \begin{cases} \frac{1}{3}, & \alpha = 0 \\ \frac{1}{2\alpha^3}\left[\ln\left(\frac{1+\alpha}{1-\alpha}\right) - 2\alpha\right], & \alpha \neq 0 \end{cases}$$

MLE asymptotic Variance:

$$\text{Var}(\tilde{\alpha}) = \frac{1}{nI(\alpha)} = \begin{cases} \frac{3}{n}, & \alpha = 0 \\ \frac{2\alpha^3}{n\left[\ln\left(\frac{1+\alpha}{1-\alpha}\right)-2\alpha\right]}, & \alpha \neq 0 \end{cases}$$

Efficiency Comparison

$$\text{eff}(\hat{\alpha}, \hat{\alpha}_{\text{MoM}}) = \frac{\text{Var}(\hat{\alpha}_{\text{MoM}})}{\text{Var}(\hat{\alpha})} = \frac{3-\alpha^2}{4-\alpha^2}$$
$$\text{eff}(\hat{\alpha}, \hat{\alpha}_{\text{MLE}}) = \frac{\text{Var}(\hat{\alpha}_{\text{MLE}})}{\text{Var}(\hat{\alpha})} = \begin{cases} \frac{2\alpha^3}{(4-\alpha^2)[\ln(1+\alpha)-\ln(1-\alpha)-2\alpha]}, & \alpha \neq 0 \\ \frac{3}{4-\alpha^2}, & \alpha = 0 \end{cases}$$

Estimator	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\text{Eff}(\hat{\alpha}, \hat{\alpha}_{\text{MoM}})$	0.75	0.7494	0.7475	0.7442	0.7396	0.7333	0.7253	0.7151	0.7024	0.6865
$\text{Eff}(\hat{\alpha}, \hat{\alpha}_{\text{MLE}})$	0.75	0.7474	0.7393	0.7254	0.7048	0.6760	0.6371	0.5841	0.5103	0.3994

Conclusion: The estimators based on $Y_i's$ perform worse than the estimators based on $X_i's$.

53(c)

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} \cdot I_{[X_{(n)},\infty]}(\theta)$$
$$\therefore \hat{\theta}_{MLE} = ArgmaxL(\theta) = X_{(n)}$$

$$\begin{aligned} f_{X_{(n)}}(x_{(n)}) &= n[F_X(x_{(n)})]^{n-1} f_X(x_{(n)}) \\ &= n \cdot (\frac{x_{(n)}}{\theta})^{n-1} \cdot \frac{1}{\theta} \\ &= \frac{n x_{(n)}^{n-1}}{\theta^n},\ 0 \le x_{(n)} \le \theta \end{aligned}$$

$$\begin{aligned} E(\hat{\theta}_{MLE}) &= \int_0^\theta \frac{n x_{(n)}^n}{\theta^n} dx_{(n)} = \frac{n\theta}{n+1} \\ E(\hat{\theta}_{MLE}^2) &= \frac{n\theta^2}{n+2} \\ Var(\hat{\theta}_{MLE}) &= E(\hat{\theta}_{MLE}^2) - [E(\hat{\theta}_{MLE})]^2 = \frac{n\theta^2}{(n+2)(n+1)^2} \\ Bias(\hat{\theta}_{MLE}) &= E(\hat{\theta}_{MLE}) - \theta = \frac{-\theta}{n+1} \\ MSE(\hat{\theta}_{MLE}) &= Var(\hat{\theta}_{MLE}) + [Bias(\hat{\theta}_{MLE})]^2 = \frac{2\theta^2}{(n+2)(n+1)} \end{aligned}$$

$$\mu_1 = E(X) = \frac{\theta}{2} \Rightarrow \hat{\theta}_{MME} = 2\bar{X}$$

$$\begin{aligned} E(\hat{\theta}_{MME}) &= \theta \\ Var(\hat{\theta}_{MME}) &= \frac{\theta^2}{3n} \\ Bias(\hat{\theta}_{MME}) &= 0 \\ MSE(\hat{\theta}_{MME}) &= \frac{\theta^2}{3n} \end{aligned}$$

Variance：

$Var(\hat{\theta}_{MME}) > Var(\hat{\theta}_{MLE}),\ \forall n$

Bias：

- MME 是不偏估計

- MLE 有輕微偏差

MSE：

當 $n = 1, 2$ 時，MME 和 MLE 的 MSE 是一樣的，但當 n 很大時，MLE 的 MSE 會小於 MME 的 MSE，因此，我們可以得知當樣本數夠大時，MLE估計會比MME估計更有效。

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a

$$\because \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$
$$\therefore E(\frac{(n-1)s^2}{\sigma^2}) = n-1,\ Var(\frac{(n-1)s^2}{\sigma^2}) = 2(n-1)$$
$$\Rightarrow E(s^2) = \sigma^2,\ Var(s^2) = \frac{2\sigma^4}{n-1}$$

$$\hat{\sigma}^2 = \frac{n-1}{n}s^2$$
$$\Rightarrow E(\hat{\sigma}^2) = \frac{n-1}{n}\sigma^2,\ Var(\hat{\sigma}^2) = \frac{2(n-1)\sigma^4}{n^2}$$

所以可得知 s^2 是不偏估計。

b

$$MSE(s^2) = Var(s^2) + [Bias(s^2)]^2 = \frac{2\sigma^4}{n-1}$$
$$MSE(\hat{\sigma}^2) = Var(\hat{\sigma}^2) + [Bias(\hat{\sigma}^2)]^2 = \frac{(2n-1)\sigma^4}{n^2}$$

$$\because (2n-1)(n-1) < 2n^2$$
$$\Rightarrow \frac{2n-1}{n^2} < \frac{2}{n-1}$$
$$\therefore MSE(\hat{\sigma}^2) < MSE(s^2)$$

c

$$Let\ Y = \rho\sum_{i=1}^{n}(X_i - \bar{X})^2,\ E(Y) = \rho(n-1)\sigma^2,\ Var(Y) = 2\rho^2(n-1)\sigma^4$$
$$\Rightarrow MSE(Y) = 2\rho^2(n-1)\sigma^4 + (\rho n - \rho - 1)^2\sigma^4$$

$$Let\ f(\rho) = \sigma^4[2\rho^2(n-1) + (\rho n - \rho - 1)^2]$$
$$\Rightarrow f'(\rho) = 2(n-1)\sigma^4(\rho n + \rho - 1)$$
$$and\ f''(\rho) = 2(n-1)(n+1)\sigma^4 > 0$$

$$Set\ f'(\rho) = 0 \Rightarrow \rho = \frac{1}{n+1}$$

60(e)

$$L(\tau) = \frac{1}{\tau^n} e^{\frac{-\sum_{i=1}^n x_i}{\tau}}$$
$$l(\tau) = -n\log\tau - \frac{1}{\tau}\sum_{i=1}^n x_i$$

Let $l'(\tau) = 0$, $\hat{\tau} = \bar{x}$, $l''(\tau)|_{\tau=\hat{\tau}} < 0$. Therefore, the MLE of τ is $\bar{X}$(可在HW.4的Solution中找到).

$$\bar{X} \sim \Gamma(n, \frac{\tau}{n})\text{(可在HW.4的Solution中找到)}$$
$$E(\bar{X}) = \tau\text{(可在HW.4的Solution中找到)}$$
$$Var(\bar{X}) = \frac{\tau^2}{n}$$
$$I(\tau) = -E(\frac{\partial^2 \log f(x;\tau)}{\partial\tau^2}) = -E(\frac{1}{\tau^2} + \frac{-2x}{\tau^3}) = \frac{-1}{\tau^2} + \frac{2}{\tau^2} = \frac{1}{\tau^2}$$
$$CRLB(\bar{X}) = \frac{1}{nI(\tau)} = \frac{\tau^2}{n}$$

因為 $\bar{X}$ 是不偏估計且 $Var(\bar{X}) = CRLB(\bar{X})$，所以沒有其他 unbiased estimator 會有更小的 variance。

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$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{\Gamma(\alpha)\beta^\alpha} exp((\alpha-1)logx - \frac{x}{\beta})$$

$$c_1(\alpha,\beta) = \alpha - 1,\ c_2(\alpha,\beta) = \frac{1}{\beta},\ t_1(x) = logx,\ t_2(x) = x$$

因此，$f(x)$ 屬於 2-參數的指數族。而 $(\sum_{i=1}^n \log X_i, \sum_{i=1}^n X_i)$ 是 (α,β) 的充分且完備統計量。
因為 $\prod_{i=1}^n X_i = exp(\sum_{i=1}^n \log X_i)$ 是1對1的轉換，所以 $(\prod_{i=1}^n X_i, \sum_{i=1}^n X_i)$ 也是 (α,β) 的充分且完備統計量。

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