Solution Manual **Homework 5**

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Problem 8.7

Suppose that X follows a geometric distribution,

$$P(X = k) = p(1-p)^{k-1}$$

and assume an i.i.d. sample of size n.

(c) Find the asymptotic variance of the mle. Also, obtain the Fisher information of X and identify the asymptotic sampling distribution of the MLE [Hint: the mean of geometric distribution is given in LN, ch1-6, p.2-62].

Solution:

First, we need to find the maximum likelihood estimate (MLE) of p. Given the sample $X_1, \ldots, X_n$, the likelihood function is:

$$L(p) = \prod_{i=1}^{n} p(1-p)^{X_i - 1} = p^n (1-p)^{\sum_{i=1}^n X_i - n}$$

The log-likelihood function is:

$$l(p) = \ln L(p) = n \ln p + \left(\sum_{i=1}^{n} X_i - n \right) \ln(1-p)$$

Taking the derivative with respect to p and setting it to 0 to find the MLE:

$$\frac{d}{dp} l(p) = \frac{n}{p} - \frac{\sum_{i=1}^n X_i - n}{1-p} = 0$$

Solving for p yields the MLE $\hat{p} = \frac{n}{\sum_{i=1}^n X_i} = \frac{1}{\bar{X}}$.

[註]：此處 MLE 在 HW4 第 7(b) 小題中已有求得。

Next, we find the Fisher information $I(p)$ for a single observation X: The log probability mass function is $\ln f(X|p) = \ln p + (X-1)\ln(1-p)$.

First derivative: $\frac{\partial}{\partial p} \ln f(X|p) = \frac{1}{p} - \frac{X-1}{1-p}$

Second derivative: $\frac{\partial^2}{\partial p^2} \ln f(X|p) = -\frac{1}{p^2} - \frac{X-1}{(1-p)^2}$

Based on the properties of the geometric distribution (as given in the hint), the expected value is $E[X] = \frac{1}{p}$. Therefore:

$$I(p) = -E\left[\frac{\partial^2}{\partial p^2} \ln f(X|p) \right] = \frac{1}{p^2} + \frac{E[X] - 1}{(1-p)^2} = \frac{1}{p^2} + \frac{\frac{1}{p} - 1}{(1-p)^2} = \frac{1}{p^2} + \frac{1}{p(1-p)} = \frac{1}{p^2(1-p)}$$

Thus, the Fisher information for a sample of size n is $I_n(p) = nI(p) = \frac{n}{p^2(1-p)}$.

The asymptotic variance of the MLE is the inverse of the Fisher information:

$$\text{Asymptotic Var}(\hat{p}) = \frac{1}{I_n(p)} = \frac{p^2(1-p)}{n}$$

The asymptotic sampling distribution of the MLE $\hat{p}$ is a normal distribution:

$$\hat{p} \xrightarrow{d} N\left(p, \frac{p^2(1-p)}{n} \right) \quad \text{or equivalently} \quad \frac{\sqrt{n}(\hat{p} - p)}{\sqrt{p^2(1-p)}} \xrightarrow{d} N(0,1)$$

Problem 8.16
Consider an i.i.d. sample of random variables with density function

$$f(x|\sigma) = \frac{1}{2\sigma}\exp\left(-\frac{|x|}{\sigma}\right)$$

(c) Find the asymptotic variance of the mle. Also, obtain the Fisher informatio n of the i.i.d. sample and identify the asymptotic sampling distribution of the MLE [Hint: you can use gamma pdf and gamma function given in LN, ch1-6, p.2-73&74, to find $E|X_i|$ or $E(X_i^2)$].

Solution:
First, let's find the MLE of σ. The log-likelihood function is:

$$l(\sigma) = \sum_{i=1}^{n} \ln f(X_i|\sigma) = \sum_{i=1}^{n}\left(-\ln 2 - \ln\sigma - \frac{|X_i|}{\sigma}\right) = -n\ln 2 - n\ln\sigma - \frac{1}{\sigma}\sum_{i=1}^{n}|X_i|$$

Taking the derivative with respect to σ and setting it to 0:

$$\frac{d}{d\sigma}l(\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}|X_i| = 0 \implies \hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n}|X_i|$$

[註]：此處是 HW4 第 16(b) 小題的答案。
Next, we find the Fisher information $I(\sigma)$ for a single observation X:

$$\ln f(X|\sigma) = -\ln 2 - \ln\sigma - \frac{|X|}{\sigma}$$

$$\frac{\partial}{\partial\sigma}\ln f(X|\sigma) = -\frac{1}{\sigma} + \frac{|X|}{\sigma^2}$$

$$\frac{\partial^2}{\partial\sigma^2}\ln f(X|\sigma) = \frac{1}{\sigma^2} - \frac{2|X|}{\sigma^3}$$

According to the hint, we can use the Gamma pdf to find $E[|X|]$:

$$E[|X|] = \int_{-\infty}^{\infty}|x|\frac{1}{2\sigma}e^{-|x|/\sigma}dx = 2\int_0^{\infty}x\frac{1}{2\sigma}e^{-x/\sigma}dx = \int_0^{\infty}x\frac{1}{\sigma}e^{-x/\sigma}dx = \frac{\Gamma(2)}{1/\sigma}\int_0^{\infty}\underbrace{\frac{(1/\sigma)^2}{\Gamma(2)}x^{2-1}e^{-\frac{x}{\sigma}}}_{\text{pdf of }\Gamma\left(2,\frac{1}{\sigma}\right)}dx = \sigma$$

On the other hand , this is exactly the expected value of an exponential distribution with parameter σ (or $\Gamma(1,\sigma)$), so $E[|X|] = \sigma$.

$$I(\sigma) = -E\left[\frac{1}{\sigma^2} - \frac{2|X|}{\sigma^3}\right] = -\frac{1}{\sigma^2} + \frac{2E[|X|]}{\sigma^3} = -\frac{1}{\sigma^2} + \frac{2\sigma}{\sigma^3} = \frac{1}{\sigma^2}$$

The Fisher information for the i.i.d. sample is:

$$I_n(\sigma) = nI(\sigma) = \frac{n}{\sigma^2}$$

The asymptotic variance of the MLE is $[I_n(\sigma)]^{-1} = \frac{\sigma^2}{n}$.

Therefore, the asymptotic sampling distribution of the MLE $\hat{\sigma}$ is:

$$\hat{\sigma} \xrightarrow{d} N\left(\sigma, \frac{\sigma^2}{n}\right) \quad \text{or equivalently} \quad \frac{\sqrt{n}(\hat{\sigma}-\sigma)}{\sigma} \xrightarrow{d} N(0,1)$$

Problem 8.47
The Pareto distribution has been used in economics as a model for a density function with a slowly decaying tail:

$$f(x|x_0,\theta) = \theta x_0^\theta x^{-\theta-1}, \quad x \geq x_0, \quad \theta > 1$$

Assume that $x_0 > 0$ is given and that $X_1, X_2, \ldots, X_n$ is an i.i.d. sample.
(a) Find the method of moments estimate of θ.
(b) Find the mle of θ.
(c) Find the asymptotic variance of the mle. Also, obtain the Fisher information of the i.i.d. sample and identify the asymptotic sampling distribution of the MLE.

Solution:

Part (a):

First, we calculate the first moment (expected value) of the Pareto distribution:

$$E[X] = \int_{x_0}^{\infty} x\theta x_0^\theta x^{-\theta-1} dx = \theta x_0^\theta \int_{x_0}^{\infty} x^{-\theta} dx = \theta x_0^\theta \left[\frac{x^{-\theta+1}}{-\theta+1}\right]_{x_0}^{\infty}$$

Since $\theta > 1$, as $x \to \infty$, $x^{-\theta+1} \to 0$.

$$\mu_1 = E[X] = \theta x_0^\theta \left(0 - \frac{x_0^{1-\theta}}{1-\theta}\right) = \frac{\theta x_0}{\theta - 1} \Rightarrow \theta = \frac{\mu_1}{\mu_1 - x_0}$$

Setting the sample mean $\bar{X}$ to estimate μ_1,thus the method of moments (MOM) estimate is:

$$\tilde{\theta} = \frac{\bar{X}}{\bar{X} - x_0}$$

Part (b):

Given the sample $X_1, \ldots, X_n$, the likelihood function is:

$$L(\theta) = \prod_{i=1}^{n} \theta x_0^\theta X_i^{-\theta-1} = \theta^n x_0^{n\theta} \left(\prod_{i=1}^{n} X_i\right)^{-\theta-1}$$

The log-likelihood function is:

$$l(\theta) = \ln L(\theta) = n\ln\theta + n\theta \ln x_0 - (\theta+1)\sum_{i=1}^{n} \ln X_i$$

Taking the derivative with respect to θ and setting it to 0:

$$\frac{d}{d\theta} l(\theta) = \frac{n}{\theta} + n\ln x_0 - \sum_{i=1}^{n} \ln X_i = 0$$

$$\frac{n}{\theta} = \sum_{i=1}^{n} \ln X_i - n\ln x_0 = \sum_{i=1}^{n} (\ln X_i - \ln x_0) = \sum_{i=1}^{n} \ln\left(\frac{X_i}{x_0}\right)$$

Solving for θ, the maximum likelihood estimate (MLE) is:

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \ln(X_i/x_0)}$$

Check $\frac{d^2}{d\theta^2} l(\theta)\Big|_{\theta=\hat{\theta}} < 0$:

$$\frac{d^2}{d\theta^2} l(\theta)\Big|_{\theta=\hat{\theta}} = -\frac{n}{\hat{\theta}^2} < 0 \quad (\because n > 0, \hat{\theta} > 0) \quad \therefore \hat{\theta} \text{ is MLE.}$$

Part (c):
Calculate the Fisher information $I(\theta)$ for a single observation X:

$$\ln f(X|\theta) = \ln\theta + \theta \ln x_0 - (\theta+1)\ln X$$

$$\frac{\partial}{\partial\theta}\ln f(X|\theta) = \frac{1}{\theta} + \ln x_0 - \ln X$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(X|\theta) = -\frac{1}{\theta^2}$$

Since the second derivative is a constant (does not depend on X), we have:

$$I(\theta) = -E\left[-\frac{1}{\theta^2}\right] = \frac{1}{\theta^2}$$

The Fisher information for the i.i.d. sample is:

$$I_n(\theta) = nI(\theta) = \frac{n}{\theta^2}$$

The asymptotic variance of the MLE is $[I_n(\theta)]^{-1} = \frac{\theta^2}{n}$.

The asymptotic sampling distribution of the MLE $\hat{\theta}$ is:

$$\hat{\theta} \xrightarrow{d} N\left(\theta, \frac{\theta^2}{n}\right) \quad \text{or equivalently} \quad \frac{\sqrt{n}(\hat{\theta}-\theta)}{\theta} \xrightarrow{d} N(0,1)$$

Question 50(c)

Objective: Find the asymptotic variance of the MLE(We have solved it in HW4 50(b)), obtain the Fisher information of the i.i.d. sample, and identify the asymptotic sampling distribution of the MLE.

Step 1: Log-likelihood function and its derivatives

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Rayleigh}(\theta)$. The probability density function (pdf) is:

$$f(x|\theta) = \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, \quad x \geq 0, \theta > 0$$

The likelihood function for the sample is:

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \left(\frac{x_i}{\theta^2} e^{-x_i^2/(2\theta^2)} \right) = \frac{\prod_{i=1}^{n} x_i}{\theta^{2n}} \exp\left(-\frac{1}{2\theta^2} \sum_{i=1}^{n} x_i^2 \right)$$

The log-likelihood function $l(\theta)$ is:

$$l(\theta) = \ln L(\theta) = \sum_{i=1}^{n} \ln(x_i) - 2n \ln(\theta) - \frac{1}{2\theta^2} \sum_{i=1}^{n} x_i^2$$

Taking the first derivative with respect to θ:

$$\frac{\partial l(\theta)}{\partial \theta} = -\frac{2n}{\theta} + \frac{1}{\theta^3} \sum_{i=1}^{n} x_i^2$$

Taking the second derivative with respect to θ:

$$\frac{\partial^2 l(\theta)}{\partial \theta^2} = \frac{2n}{\theta^2} - \frac{3}{\theta^4} \sum_{i=1}^{n} x_i^2$$

Step 2: Calculate $E[X_i^2]$ using the Gamma function (as per the hint)

To find the Fisher information, we first need to compute $E[X^2]$ for a single observation:

$$E[X^2] = \int_0^\infty x^2 f(x|\theta) dx = \int_0^\infty x^2 \left(\frac{x}{\theta^2} e^{-x^2/(2\theta^2)} \right) dx = \int_0^\infty \frac{x^3}{\theta^2} e^{-x^2/(2\theta^2)} dx$$

Let $y = \frac{x^2}{2\theta^2}$. Then $x^2 = 2\theta^2 y$, and by differentiating, we get $2xdx = 2\theta^2 dy \implies xdx = \theta^2 dy$. When $x = 0$, we have $y = 0$; when $x \to \infty$, we have $y \to \infty$. Substituting these into the integral:

$$E[X^2] = \int_0^\infty \left(\frac{x^2}{\theta^2} \right) e^{-x^2/(2\theta^2)} (xdx) = \int_0^\infty \left(\frac{2\theta^2 y}{\theta^2} \right) e^{-y} (\theta^2 dy)$$

$$E[X^2] = 2\theta^2 \int_0^\infty y e^{-y} dy = 2\theta^2 \int_0^\infty y^{2-1} e^{-y} dy$$

Recall the definition of the Gamma function: $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$. Therefore, the integral is exactly $\Gamma(2)$.

$$E[X^2] = 2\theta^2 \cdot \Gamma(2) = 2\theta^2 \cdot 1! = 2\theta^2$$

Step 3: Fisher Information of the i.i.d. sample

The Fisher Information of the sample, $I_n(\theta)$, is defined as:

$$I_n(\theta) = -E\left[\frac{\partial^2 l(\theta)}{\partial \theta^2}\right] = -E\left[\frac{2n}{\theta^2} - \frac{3}{\theta^4}\sum_{i=1}^{n} X_i^2\right]$$

By linearity of expectation:

$$I_n(\theta) = -\frac{2n}{\theta^2} + \frac{3}{\theta^4}\sum_{i=1}^{n} E[X_i^2]$$

Substitute $E[X_i^2] = 2\theta^2$ into the equation:

$$I_n(\theta) = -\frac{2n}{\theta^2} + \frac{3}{\theta^4} \cdot n \cdot (2\theta^2) = -\frac{2n}{\theta^2} + \frac{6n}{\theta^2} = \frac{4n}{\theta^2}$$

Step 4: Asymptotic Variance and Sampling Distribution
The asymptotic variance of the Maximum Likelihood Estimator ($\hat{\theta}_{MLE}$) is the inverse of the Fisher Information of the sample:

$$\text{Asymptotic Var}(\hat{\theta}_{MLE}) = [I_n(\theta)]^{-1} = \frac{\theta^2}{4n}$$

By the properties of maximum likelihood estimators, under regularity conditions, $\hat{\theta}_{MLE}$ is asymptotically normally distributed. Therefore, the asymptotic sampling distribution of the MLE is:

$$\hat{\theta}_{MLE} \xrightarrow{d} N\left(\theta, \frac{\theta^2}{4n}\right)$$

Alternatively, this can be written as:

$$\frac{\sqrt{n}(\hat{\theta}_{MLE} - \theta)}{\theta/2} \xrightarrow{d} N(0,1)$$

Question 58

Let X_1, X_2, X_3 be the observed counts for genotypes Hp1-1 (AA), Hp1-2 (Aa), and Hp2-2 (aa) respectively. The sample size is $n = 190$. From the given data, we have $x_1 = 10, x_2 = 68, x_3 = 112$. The probabilities for each genotype are $p_1 = (1-\theta)^2$, $p_2 = 2\theta(1-\theta)$, and $p_3 = \theta^2$. The joint distribution of (X_1, X_2, X_3) is a multinomial distribution.

(a) Find the mle of θ.

The likelihood function is given by:

$$L(\theta) \propto p_1^{x_1} p_2^{x_2} p_3^{x_3} = \left[(1-\theta)^2\right]^{x_1} \left[2\theta(1-\theta)\right]^{x_2} \left[\theta^2\right]^{x_3}$$

$$L(\theta) \propto (1-\theta)^{2x_1} \cdot \theta^{x_2}(1-\theta)^{x_2} \cdot \theta^{2x_3} = \theta^{x_2+2x_3}(1-\theta)^{2x_1+x_2}$$

The log-likelihood function is:

$$l(\theta) = \ln L(\theta) = C + (x_2 + 2x_3)\ln(\theta) + (2x_1 + x_2)\ln(1-\theta)$$

where C is a constant independent of θ.

To find the MLE, we take the first derivative of $l(\theta)$ with respect to θ and set it to zero:

$$\frac{\partial l(\theta)}{\partial \theta} = \frac{x_2 + 2x_3}{\theta} - \frac{2x_1 + x_2}{1-\theta} = 0$$

Solving for θ:

$$\begin{aligned}
\frac{x_2 + 2x_3}{\theta} &= \frac{2x_1 + x_2}{1-\theta} \\
(x_2 + 2x_3)(1-\theta) &= (2x_1 + x_2)\theta \\
x_2 + 2x_3 - (x_2 + 2x_3)\theta &= (2x_1 + x_2)\theta \\
x_2 + 2x_3 &= \theta(2x_1 + x_2 + x_2 + 2x_3) \\
x_2 + 2x_3 &= \theta(2x_1 + 2x_2 + 2x_3) = \theta[2(x_1 + x_2 + x_3)] = 2n\theta
\end{aligned}$$

Thus, the Maximum Likelihood Estimator (MLE) of θ is:

$$\hat{\theta} = \frac{X_2 + 2X_3}{2n}$$

Note: The MLE above we have derived in Example 6.15 of Lecture Note p.3-24$\sim$ 25.

Substitute the observed data ($x_1 = 10, x_2 = 68, x_3 = 112, n = 190$):

$$\hat{\theta} = \frac{68 + 2(112)}{2(190)} = \frac{68 + 224}{380} = \frac{292}{380} \approx 0.7684$$

(b) Find the asymptotic variance of the mle and identify the asymptotic sampling distribution.

First, we find the second derivative of the log-likelihood function:

$$\frac{\partial^2 l(\theta)}{\partial \theta^2} = -\frac{X_2 + 2X_3}{\theta^2} - \frac{2X_1 + X_2}{(1-\theta)^2}$$

As per the hint, the marginal distributions are binomial: $X_1 \sim B(n, (1-\theta)^2)$, $X_2 \sim B(n, 2\theta(1-\theta))$, and $X_3 \sim B(n, \theta^2)$. Therefore, the expected values are:

$$\begin{aligned}
E[X_1] &= n(1-\theta)^2 \\
E[X_2] &= 2n\theta(1-\theta) \\
E[X_3] &= n\theta^2
\end{aligned}$$

The Fisher Information $I_n(\theta)$ is:

$$\begin{aligned}
I_n(\theta) &= -E\left[\frac{\partial^2 l(\theta)}{\partial\theta^2}\right] = E\left[\frac{X_2+2X_3}{\theta^2}+\frac{2X_1+X_2}{(1-\theta)^2}\right] \\
&= \frac{E[X_2]+2E[X_3]}{\theta^2}+\frac{2E[X_1]+E[X_2]}{(1-\theta)^2} \\
&= \frac{2n\theta(1-\theta)+2n\theta^2}{\theta^2}+\frac{2n(1-\theta)^2+2n\theta(1-\theta)}{(1-\theta)^2} \\
&= \frac{2n\theta(1-\theta+\theta)}{\theta^2}+\frac{2n(1-\theta)(1-\theta+\theta)}{(1-\theta)^2} \\
&= \frac{2n\theta}{\theta^2}+\frac{2n(1-\theta)}{(1-\theta)^2} \\
&= \frac{2n}{\theta}+\frac{2n}{1-\theta} = \frac{2n(1-\theta)+2n\theta}{\theta(1-\theta)} = \frac{2n}{\theta(1-\theta)}
\end{aligned}$$

The asymptotic variance of the MLE is the inverse of the Fisher Information:

$$\text{Asymptotic Var}(\hat{\theta}) = \frac{1}{I_n(\theta)} = \frac{\theta(1-\theta)}{2n}$$

Using the estimated $\hat{\theta} \approx 0.7684$:

$$\text{Asymptotic Var}(\hat{\theta}) \approx \frac{0.7684(1-0.7684)}{2(190)} \approx \frac{0.7684\times 0.2316}{380} \approx 0.000468$$

Finally, by the properties of MLE, the asymptotic sampling distribution is normal:

$$\hat{\theta} \xrightarrow{d} N\left(\theta, \frac{\theta(1-\theta)}{2n}\right)$$

or equivalently,

$$\frac{\sqrt{n}\left(\hat{\theta}-\theta\right)}{\sqrt{\theta(1-\theta)/2}} \xrightarrow{d} N(0,1)$$