Ch8

7.

(a)

By LN Ch1-b P.62.

$\mu_1 = E(X) = \frac{1}{p} \quad p = \frac{1}{\mu_1}$

Therefore. MME of p is $\frac{1}{\bar{X}}$

(b) $X_1, \dots X_n \overset{iid}{\sim} Geo(p)$

$L(p|\underline{X}) = \prod_{i=1}^{n} p(1-p)^{x_i-1}$

$= p^n (1-p)^{\sum_{i=1}^{n} x_i - n}$

$l(p|\underline{X}) = n \log p + \left(\sum_{i=1}^{n} x_i - n\right) \log(1-p)$

$\frac{\partial l(p|\underline{X})}{\partial p} = \frac{n}{p} - \frac{\sum_{i=1}^{n} x_i - n}{1-p} \overset{\text{set}}{=} 0$

$\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i}$

$\left.\frac{\partial^2 l(p|\underline{X})}{\partial p^2}\right|_{p=\hat{p}} = -\frac{n}{\hat{p}^2} - \frac{\sum_{i=1}^{n} x_i - n}{(1-\hat{p})^2} = \frac{-\left(\sum_{i=1}^{n} x_i\right)^3}{n\left(\sum_{i=1}^{n} x_i - n\right)} < 0$

$\therefore \hat{p}_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$

9. $X_1 \cdots X_n \overset{iid}{\sim} Poi(\lambda)$

$\hat{\lambda} = \bar{X} = \sum_{i=1}^{n} X_i$ is a random variable (an estimator 估計式)

$\hat{\lambda}_0 = \bar{x} = \sum_{i=1}^{n} x_i$ is a value (an estimate 估計值), where

$x_1, x_2, \dots, x_n$ is a realization of $X_1 \cdots X_n$

所以24.9是估計式 $\hat{\lambda}$ 產出的一个估計值

(i.e. the estimate $\lambda_0 = 24.9$ is a realization of the estimator $\hat{\lambda}$)

16.

(a) $f(x|\sigma) = \frac{1}{2\sigma} e^{\frac{-|x|}{\sigma}}$

$$\mu_1 = E(X) = \int_{-\infty}^{\infty} x \frac{1}{2\sigma} e^{\frac{-|x|}{\sigma}} dx$$

$$= \int_0^{\infty} \frac{x}{2\sigma} e^{\frac{-x}{\sigma}} dx + \int_{-\infty}^{0} \frac{x}{2\sigma} e^{\frac{x}{\sigma}} dx \quad \text{let } t = -x$$

$$= \int_0^{\infty} \frac{x}{2\sigma} e^{\frac{-x}{\sigma}} dx - \int_0^{\infty} \frac{t}{2\sigma} e^{\frac{-t}{\sigma}} dt = 0$$

$$\mu_2 = E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2\sigma} e^{\frac{-|x|}{\sigma}} dx$$

$$= \int_0^{\infty} x^2 \frac{1}{2\sigma} e^{\frac{-x}{\sigma}} dx + \int_{-\infty}^{0} x^2 \frac{1}{2\sigma} e^{\frac{x}{\sigma}} dx \quad \text{let } t = -x$$

$$= \int_0^{\infty} x^2 \cdot \frac{1}{2\sigma} e^{\frac{-x}{\sigma}} dx + \int_0^{\infty} t^2 \frac{1}{2\sigma} e^{\frac{-t}{\sigma}} dt$$

$$= 2\int_0^{\infty} x^2 \frac{1}{2\sigma} e^{\frac{-x}{\sigma}} dx$$

$$= \frac{2}{2\sigma} \int_0^{\infty} x^2 e^{\frac{-x}{\sigma}} dx$$

$$= \frac{1}{\sigma} \int_0^{\infty} \sigma^2 y^2 e^{-y} \sigma dy \quad \text{let } y = \frac{x}{\sigma},\ \frac{dy}{dx} = \frac{1}{\sigma},\ x = \sigma y$$

$$= \sigma^2 \int_0^{\infty} y^2 e^{-y} dy \quad \rightarrow \text{Gamma function}$$

$$= \Gamma(3)\sigma^2 = 2\sigma^2$$

$\mu_2 = 2\sigma^2$ $\quad \sigma = \sqrt{\frac{1}{2}\mu_2}$ $\quad \therefore$ the MME of σ is $\sqrt{\frac{\sum_{i=1}^{n} X_i^2}{2n}}$

(b)

$$L(\sigma|\underline{x}) = \prod_{i=1}^{n} \frac{1}{2\sigma} e^{\frac{-|x_i|}{\sigma}}$$

$$= (2\sigma)^{-n} e^{\frac{-\sum_{i=1}^{n}|x_i|}{\sigma}}$$

$$\ell(\sigma|\underline{x}) = -n\log 2\sigma - \frac{\sum_{i=1}^{n}|x_i|}{\sigma}$$

$$\frac{\partial \ell(\sigma|\underline{x})}{\partial \sigma} = \frac{-n}{\sigma} + \frac{\sum_{i=1}^{n}|x_i|}{\sigma^2} \overset{\text{set}}{=} 0$$

$$\hat{\sigma} = \frac{\sum_{i=1}^{n}|x_i|}{n}$$

$$\left.\frac{\partial^2 \ell(\sigma|\underline{x})}{\partial \sigma^2}\right|_{\sigma=\hat{\sigma}} = \frac{n}{\hat{\sigma}^2} - \frac{2\sum_{i=1}^{n}|x_i|}{\hat{\sigma}^3} = \frac{-n^3}{\sum_{i=1}^{n}|x_i|} < 0$$

$$\therefore \hat{\sigma}_{MLE} = \frac{\sum_{i=1}^{n}|x_i|}{n}$$

19. $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$L(\mu, \sigma|\underline{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x_i-\mu)^2}{2\sigma^2}}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{\frac{-\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}$$

$$\ell(\mu, \sigma|\underline{x}) = -\frac{n}{2}\log 2\pi - n\log\sigma - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$$

(a) When μ is known

$$\frac{\partial \ell(\mu, \sigma|\underline{x})}{\partial \sigma} = \frac{-n}{\sigma} + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^3} \overset{\text{set}}{=} 0$$

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\left.\frac{\partial^2 \ell(\mu, \sigma|\underline{x})}{\partial \sigma^2}\right|_{\sigma=\hat{\sigma}} = \frac{n}{\hat{\sigma}^2} - \frac{3\sum_{i=1}^{n}(x_i-\mu)^2}{\hat{\sigma}^4}$$

$$= \frac{1}{\hat{\sigma}^2}(n-3n) < 0 \qquad \therefore \hat{\sigma}_{MLE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$$

(b)

When σ is known

$$\frac{\partial \ell(\mu,\sigma|\underline{x})}{\partial \mu} = \frac{2\sum_{i=1}^{n}(x_i-\mu)}{2\sigma^2} \overset{\text{set}}{=} 0$$

$$\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\left.\frac{\partial^2 \ell(\mu,\sigma|\underline{x})}{\partial \mu^2}\right|_{\mu=\hat{\mu}} = -\frac{n}{\sigma^2} < 0$$

$$\therefore \hat{\mu}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

21. $X_1 \cdots X_n \overset{iid}{\sim} f(x|\theta) = e^{-(x-\theta)} \quad x \geq \theta$

(a)

Let $Y = X-\theta \Rightarrow X = Y+\theta \quad |J| = 1$

$\therefore f(y|\theta) = e^{-y} \quad y \geq 0$

$Y \sim Exp(1)$

$E(Y) = 1 = E(X-\theta) \quad \because E(X) = \mu_1 = 1+\theta$

$\Rightarrow \theta = \mu_1 - 1$

$\Rightarrow \hat{\theta}_{MME} = \bar{X} - 1$

(b)

$$L(\theta|\underline{x}) = \prod_{i=1}^{n} e^{-(x_i-\theta)} I_{[x \geq \theta]}$$

$$= e^{-\sum_{i=1}^{n} x_i + n\theta} I_{\{x_{(1)} \geq \theta\}} \qquad x_{(1)} = \min\{x_1, \cdots, x_n\}$$

$\because L(\theta)$ is an increasing positive function of θ when $\theta \leq X_{(1)}$ and $L(\theta) = 0$ when $\theta > X_{(1)}$

$\therefore L(\theta)$ reaches its maximum when $\theta = X_{(1)} \Rightarrow \hat{\theta}_{MLE} = X_{(1)}$

Ch8:

#26

Assumption: ① sampling without replacement

② when sampling, each animal will be draw out at same probability

③ There is no trap-happy or trap-shy effect. That is, previously captured animal do not become more or less likely to be captured again

N: population size

T: number of tagged animal

n: sample size

X: random variable denote the number of tagged animal in the sample.

$X \sim$ Hypergeometric $(N\ T.\ n)$

$$L(N|x) = \frac{\binom{T}{x}\binom{N-T}{n-x}}{\binom{N}{n}}$$

Find $\hat{N}_{MLE} \Rightarrow \hat{N}_{MLE} = \operatorname{argmax} L(N|x)$

Let $\frac{L(N|x)}{L(N-1|x)} \geq 1$

$$\Rightarrow \frac{\binom{T}{x}\binom{N-T}{n-x}}{\binom{N}{n}} \cdot \frac{\binom{N-1}{n}}{\binom{T}{x}\binom{N-T-1}{n-x}} = \frac{\frac{(N-T)!}{(n-x)!(N-T-n+x)!}\frac{(N-1)!}{n!(N-1-n)!}}{\frac{N!}{n!(N-n)!}\frac{(N-T-1)!}{(n-x)!(N-T-n+x-1)!}} = \frac{(N-T)(N-n)}{N(N-T-n+x)} \geq 1 \Rightarrow \hat{N}_{MLE} \leq \frac{Tn}{x}$$

Let $\frac{L(N|x)}{L(N+1|x)} \geq 1 \Rightarrow \frac{\binom{T}{x}\binom{N-T}{n-x}}{\binom{N}{n}} \cdot \frac{\binom{N+1}{n}}{\binom{T}{x}\binom{N+1-T}{n-x}} = \frac{(N+1)(N+1-T-n+x)}{(N+1-n)(N+1-T)} \geq 1 \Rightarrow \hat{N}_{MLE} \geq \frac{Tn}{x} - 1$

$\Rightarrow \hat{N}_{MLE}$ is an integer satisfying $\frac{Tn}{x} - 1 \leq \hat{N}_{MLE} \leq \frac{Tn}{x}$

In the problem. $T = 100$. $n = 50$. $x = 20$

$$\frac{100 \cdot 50}{20} - 1 \leq \hat{N}_{MLE} \leq \frac{100 \cdot 50}{20} \Rightarrow 249 \leq \hat{N}_{MLE} \leq 250 \Rightarrow \hat{N}_{MLE} = 249 \text{ or } 250.$$

#50

(a) $f(x|\theta)=\begin{cases}\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}} & x\geq 0\\ 0 & \text{o.w.}\end{cases}$

$\mu_1=E(X)=\int_0^\infty x\cdot\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}dx \overset{(*)}{=}\int_0^\infty \frac{y}{\theta^2}e^{-\frac{y}{2\theta^2}}\cdot\frac{1}{2\sqrt{y}}dy=\int_0^\infty \frac{1}{2\theta^2}y^{\frac{1}{2}}e^{-\frac{y}{2\theta^2}}dy$

$=\frac{1}{2\theta^2}\Gamma(\frac{3}{2})(2\theta^2)^{\frac{3}{2}}\boxed{\left[\int_0^\infty \frac{y^{\frac{3}{2}-1}e^{-\frac{y}{2\theta^2}}}{\Gamma(\frac{3}{2})(2\theta^2)^{\frac{3}{2}}}dy\right]}=1$

pdf of $Y\sim\Gamma(\frac{3}{2},2\theta^2)$

$=\sqrt{\frac{\pi}{2}}\theta$

$\Rightarrow\theta=\sqrt{\frac{2}{\pi}}\mu_1$

$\Rightarrow$ The MME of θ is $\sqrt{\frac{2}{\pi}}\cdot\frac{1}{n}\sum_{i=1}^n X_i=\sqrt{\frac{2}{\pi}}\bar{X}$

$(*)\ y=x^2\Rightarrow dy=2xdx\Rightarrow dx=\frac{1}{2\sqrt{y}}dy$

(b) $L(\theta|x_1,\cdots,x_n)=\prod_{i=1}^n\frac{x_i}{\theta^2}e^{-\frac{x_i^2}{2\theta^2}}=\theta^{-2n}\cdot(\prod_{i=1}^n x_i)\cdot\exp\{-\frac{\sum_{i=1}^n x_i^2}{2\theta^2}\}$

$\ell(\theta|x_1,\cdots,x_n)=\log(L(\theta|x_1,\cdots,x_n))=-2n\log\theta+\sum_{i=1}^n\log(x_i)-\frac{\sum_{i=1}^n x_i^2}{2\theta^2}$

$\frac{\partial}{\partial\theta}\ell(\theta|x_1,\cdots,x_n)=-2n\frac{1}{\theta}+\frac{\sum_{i=1}^n x_i^2}{\theta^3}\overset{\text{set}}{=}0\Rightarrow\hat{\theta}_{MLE}=\sqrt{\frac{\sum_{i=1}^n x_i^2}{2n}}$

$\frac{\partial^2}{\partial\theta^2}\ell(\theta|x_1,\cdots,x_n)\Big|_{\theta=\hat{\theta}_{MLE}}=(\frac{2n}{\theta^2}-\frac{3\sum_{i=1}^n x_i^2}{\theta^4})\Big|_{\theta=\hat{\theta}_{MLE}}=\frac{2n\cdot 2n}{\sum_{i=1}^n x_i^2}-\frac{(2n)^2\cdot 3\cdot\sum_{i=1}^n x_i^2}{(\sum_{i=1}^n x_i^2)^2}=\frac{-8n^2}{(\sum_{i=1}^n x_i^2)}<0$

$\Rightarrow\hat{\theta}_{MLE}=\sqrt{\frac{\sum_{i=1}^n x_i^2}{2n}}$

#5 $L(\theta|x_1,\cdots,x_n)=\prod_{i=1}^{2m+1}(\frac{1}{2}e^{-|x_i-\theta|})=\frac{1}{2^{2m+1}}\exp\{-\sum_{i=1}^{2m+1}|x_i-\theta|\}$

$\max_{-\infty<\theta<\infty}\{L(\theta|x_1,\cdots,x_n)\}\Leftrightarrow\max_{-\infty<\theta<\infty}\{\exp\{-\sum_{i=1}^{2m+1}|x_i-\theta|\}\Leftrightarrow\min_{-\infty<\theta<\infty}\{\sum_{i=1}^{2m+1}|x_i-\theta|\}$

$\Leftrightarrow\min_{-\infty<\theta<\infty}\{\sum_{i=1}^{2m+1}|x_{(i)}-\theta|\}$

① $x_{(m)}\le\theta\le x_{(m+2)}\Rightarrow\sum_{i=1}^{2m+1}|x_{(i)}-\theta|=\sum_{i=1}^{m}(\theta-x_{(i)})+\sum_{i=m+2}^{2m+1}(x_{(i)}-\theta)+|x_{(m+1)}-\theta|$

$=\sum_{i=1}^{m}-x_{(i)}+\sum_{i=m+2}^{2m+1}x_{(i)}+|x_{(m+1)}-\theta|$

$\Rightarrow$ when $\theta=x_{(m+1)}$, $\sum_{i=1}^{2m+1}|x_{(i)}-\theta|$ has minimum $\sum_{i=1}^{m}-x_{(i)}+\sum_{i=m+2}^{2m+1}x_{(i)}$

② $x_{(m+2)}<\theta\le x_{(m+3)}\Rightarrow L(\theta)=\sum_{i=1}^{2m+1}|x_{(i)}-\theta|=\sum_{i=1}^{m+2}(\theta-x_{(i)})+\sum_{i=m+3}^{2m+1}(x_{(i)}-\theta)$

$\frac{\partial L(\theta)}{\partial\theta}=(m+2)-(m-1)=3>0$

$\Rightarrow L(\theta)$ 在 $(x_{(m+2)},x_{(m+3)}]$ 這個區間是遞增

③ $x_{(m-1)}\le\theta<x_{(m)}\Rightarrow L(\theta)=\sum_{i=1}^{2m+1}|x_{(i)}-\theta|=\sum_{i=1}^{m-1}(\theta-x_{(i)})+\sum_{i=m}^{2m+1}(x_{(i)}-\theta)$

$\frac{\partial L(\theta)}{\partial\theta}=(m-1)-(m+2)=-3<0$

$\Rightarrow L(\theta)$ 在 $[x_{(m-1)},x_{(m)})$ 這個區間是遞減

④ $x_{(j-1)}\le\theta<x_{(j)}$, $j=2,\cdots,m-1\Rightarrow L(\theta)=\sum_{i=1}^{2m+1}|x_{(i)}-\theta|=\sum_{i=1}^{j-1}(\theta-x_{(i)})+\sum_{i=j}^{2m+1}(x_{(i)}-\theta)$

$\frac{\partial L(\theta)}{\partial\theta}=(j-1)-(2m+1-j+1)=2j-2m-3<0\ (j<m)$

$\Rightarrow L(\theta)$ 在 $[x_{(j-1)},x_{(j)})$ 這個區間是遞減, $j=2,3,\cdots,m-1$

⑤ $x_{(j-1)}<\theta\le x_{(j)}$, $j=m+4,\cdots,2m+1\Rightarrow L(\theta)=\sum_{i=1}^{2m+1}|x_{(i)}-\theta|=\sum_{i=1}^{j-1}(\theta-x_{(i)})+\sum_{i=j}^{2m+1}(x_{(i)}-\theta)$

$\frac{\partial L(\theta)}{\partial\theta}=(j-1)-(2m+1-j+1)=2j-2m-3>0\ (j>m+3)$

$\Rightarrow L(\theta)$ 在 $(x_{(j-1)},x_{(j)}]$ 這個區間是遞增, $j=m+4,\cdots,2m+1$

⑥ $\theta<x_{(1)}\Rightarrow L(\theta)=\sum_{i=1}^{2m+1}|x_{(i)}-\theta|=\sum_{i=1}^{2m+1}(x_{(i)}-\theta)$, $\frac{\partial L(\theta)}{\partial\theta}=-(2m+1)<0$

$\Rightarrow L(\theta)$ 在 $(-\infty,x_{(1)})$ 這個區間是遞減

⑦ $\theta>x_{(2m+1)}\Rightarrow L(\theta)=\sum_{i=1}^{2m+1}|x_{(i)}-\theta|=\sum_{i=1}^{2m+1}(\theta-x_{(i)})$, $\frac{\partial L(\theta)}{\partial\theta}=2m+1>0$

$\Rightarrow L(\theta)$ 在 $(x_{(2m+1)},\infty)$ 這個區間是遞增

$\Rightarrow$ 綜合以上①~⑦的討論可得知最小值發生在 $\theta=x_{(m+1)}\Rightarrow\hat{\theta}_{MLE}=x_{(m+1)}$

#60

(a) $L(\tau|x_1,\dots,x_n)=\prod_{i=1}^{n}\frac{1}{\tau}e^{-x_i/\tau}=(\frac{1}{\tau})^n\exp\{-\frac{\sum_{i=1}^{n}x_i}{\tau}\}$

$\ell(\tau|x_1,\dots,x_n)=\log L(\tau|x_1,\dots,x_n)=-n\log\tau-\frac{\sum_{i=1}^{n}x_i}{\tau}$

$\frac{\partial}{\partial\tau}\ell(\tau|x_1,\dots,x_n)=-\frac{n}{\tau}+\frac{\sum_{i=1}^{n}x_i}{\tau^2}\overset{set}{=}0\Rightarrow\hat{\tau}_{MLE}=\frac{1}{n}\sum_{i=1}^{n}x_i=\bar{X}$

$\frac{\partial^2}{\partial\tau^2}\ell(\tau|x_1,\dots,x_n)\Big|_{\tau=\hat{\tau}_{MLE}}=(\frac{n}{\tau^2}-\frac{2\sum_{i=1}^{n}x_i}{\tau^3})\Big|_{\tau=\hat{\tau}_{MLE}}=\frac{n}{\bar{X}^2}-\frac{2n\bar{X}}{\bar{X}^3}=-\frac{n}{\bar{X}^2}<0$

$\Rightarrow\hat{\tau}_{MLE}=\bar{X}$

(b) $X_i\overset{iid}{\sim}Exp(\frac{1}{\tau})=Gamma(1,\frac{1}{\tau})\Rightarrow\sum_{i=1}^{n}X_i\sim Gamma(n,\frac{1}{\tau})$

$\Rightarrow\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i\sim Gamma(n,\frac{n}{\tau})$

(c) $E(\bar{X})=\frac{n}{\frac{n}{\tau}}=\tau$, $Var(\bar{X})=\frac{n}{(\frac{n}{\tau})^2}=\frac{\tau^2}{n}$

By CLT, $\frac{\bar{X}-\tau}{(\frac{\tau^2}{n})^{\frac{1}{2}}}=\frac{\sqrt{n}(\bar{X}-\tau)}{\tau}\overset{d}{\sim}N(0,1)\Rightarrow\bar{X}\sim N(\tau,\frac{\tau^2}{n})$

(d) $E(\hat{\tau}_{MLE})=E(\bar{X})=\tau\Rightarrow\hat{\tau}_{MLE}$ is an unbiased estimator

$Var(\hat{\tau}_{MLE})=Var(\bar{X})=\frac{\tau^2}{n}$