Ch.5

#1

Suppose $X_i \overset{\text{indep.}}{\sim} (\mu, \sigma_i^2) \Rightarrow \bar{X} \sim (\mu, n^{-2}\sum \sigma_i^2)$

Given $\varepsilon > 0$

By Chebyshev's ineq. we have

$$P(|\bar{X} - \mu| > \varepsilon) \le \frac{Var(\bar{X})}{\varepsilon^2}$$

$$= \frac{n^{-2}\sum_{i}^{n} \sigma_i^2}{\varepsilon^2} \xrightarrow{n\to\infty} 0$$

So $\bar{X} \xrightarrow{P.} \mu$

#3

$N \sim Poi(\lambda)$

$E(N) = \lambda = 10000$

$Var(N) = \lambda = 10000$

$\Rightarrow N \overset{d}{\approx} N(10000, 100^2)$

$\Rightarrow \frac{N - 10000}{100} \overset{d}{\approx} N(0,1) = Z$

$\Rightarrow P(N > 10200)$

$= P\left(\frac{N - 10000}{100} > \frac{10200 - 10000}{100}\right)$

$\approx P(Z > 2)$

$\doteq 0.0228$ (by table)

#5

Suppose $X \sim B(n,p)$, then the mgf is

$M_X(t) = (pe^t + 1-p)^n$

Then we assume $n \to \infty$, $p \to 0$ and $np \to \lambda$

$M_X(t) = (pe^t + 1-p)^n = \left[1 + \frac{np(e^t-1)}{n}\right]^n$

$\to \exp(\lambda e^t - 1)$

(by thm. if $a_n \to a$, then $(1+\frac{a_n}{n})^n \to e^a$)

which is the mgf of $poi(\lambda)$.

By levy's continuity thm. we conclude that $B(n,p) \to Poi(\lambda)$

#10 (i) let $X_i = \begin{cases} 1, & \text{第 } i \text{ 次成功丟到 } 6 \\ 0, & \text{o/w} \end{cases}$ then $X_i \overset{i.i.d}{\sim} Ber(\frac{1}{6})$

$i = 1, \cdots, 100$, $E(X_i) = \frac{1}{6}$, $Var(X_i) = \frac{5}{36}$, and $E(\sum_1^{100} X_i) = \frac{100}{6}$, $Var(\sum_1^{100} X_i) = \frac{125}{9}$

by CLT $\sum_1^{100} X_i \xrightarrow{d} N(\frac{100}{6}, \frac{125}{9})$

$P(15 < \sum_1^{100} X_i < 20) = P\left(\frac{15-\frac{100}{6}}{\sqrt{\frac{125}{9}}} < \frac{\sum_1^{100} X_i - \frac{100}{6}}{\sqrt{\frac{125}{9}}} < \frac{20-\frac{100}{6}}{\sqrt{\frac{125}{9}}}\right)$

$\approx P\left(\frac{15-\frac{100}{6}}{\sqrt{\frac{125}{9}}} \le Z \le \frac{20-\frac{100}{6}}{\sqrt{\frac{125}{9}}}\right)$, w/ $Z \sim N(0,1)$

$\fallingdotseq 0.487$ (by table)

(ii) Let Y_i 為丟第 i 次骰子之點數

$Y_i \overset{iid}{\sim} f_Y(y) = \frac{1}{6}$, $y = 1, 2, \cdots, 6$, $i = 1, \cdots, 100$

$E(Y_i) = \frac{7}{2}$, $Var(Y_i) = \frac{35}{12}$, and $E(\sum Y_i) = 350$, $Var(\sum Y_i) = \frac{3500}{12}$

by CLT, $\sum_1^{100} Y_i \xrightarrow{d} N(350, \frac{3500}{12})$

$$P(\sum Y_i < 300) = P\left(\frac{\sum Y_i - 350}{\sqrt{3500/12}} < \frac{300-350}{\sqrt{3500/12}}\right)$$

$$\approx P\left(Z < \frac{300-350}{\sqrt{\frac{3500}{12}}}\right) \fallingdotseq 0.0017, \text{ w) } Z \sim N(0,1)$$

11 It's wrong because CLT requires each r.v. in the sequence to have a fixed distribution that doesn't change as $n \uparrow$. In this case, the poisson parameter n^{-1} change and shrink as n grows, which violates the "identically" assumption (ie., $X_{n+1}, X_{n+2}, \cdots$ must have the same distribution as $X_1, \cdots, X_n$ that form the sum $\sum_{i=1}^{n} X_i$) of the CLT.

17 because $E(\bar{X}) = \mu$, and $Var(\bar{X}) = \frac{25}{n}$,

$$0.95 = P(|\bar{X} - \mu| < 1) = P(-1 < \bar{X} - \mu < 1) = P\left(\frac{-1}{\sqrt{25/n}} < \frac{\bar{X}-\mu}{\sqrt{25/n}} < \frac{1}{\sqrt{25/n}}\right)$$

$$\overset{\text{(by CLT)}}{\approx} P\left(\frac{-1}{\sqrt{\frac{25}{n}}} < Z < \frac{1}{\sqrt{\frac{25}{n}}}\right), \text{ w) } Z \sim N(0,1)$$

$\Rightarrow \frac{1}{\sqrt{\frac{25}{n}}} = 1.96$ from normal table, $n = 96$ or 97

#18. Let X_i be the weight of ith package, $i=1,\cdots,160$

$X_i \overset{\text{indep.}}{\sim} (15, 100) \Rightarrow \sum_{i=1}^{100} X_i \sim (1500, 100^2)$

$$P(\Sigma X_i > 1700) = P\left(\frac{\Sigma X_i - 1500}{100} > \frac{1700-1500}{100}\right)$$

$$\overset{\text{(by CLT)}}{\approx} P(Z > 2) = 0.0228 \text{ (by table)}$$

, w) $Z \sim N(0,1)$

CH 5

#21.

(a) $E[\hat{I}(f)] = E\left[\frac{f(x)}{g(x)}\right] = \int_a^b \frac{f(x)}{g(x)} g(x)\,dx = \int_a^b f(x)\,dx = I(f)$

(b) $\mathrm{Var}(\hat{I}(f)) = \frac{1}{n}\mathrm{Var}\left(\frac{f(x)}{g(x)}\right) = \frac{1}{n}\left[E\left(\frac{f(x)}{g(x)}\right)^2 - \left(E\left(\frac{f(x)}{g(x)}\right)\right)^2\right]$

$= \frac{1}{n}\left[\int_a^b \frac{f^2(x)}{g^2(x)} \cdot g(x)\,dx - I^2(f)\right]$

$= \frac{1}{n}\left[\int_a^b \frac{f^2(x)}{g(x)}\,dx - I^2(f)\right]$

Assume $f(x) = \frac{1}{x^{\frac{1}{4}}} \Rightarrow f^2(x) = \frac{1}{x^{\frac{1}{2}}}$, $a=0$, $b=1$.

1° If $g \sim U(0,1)$.

$\because \int_0^1 \frac{f^2(x)}{g(x)}\,dx = \int_0^1 x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}}\Big|_0^1 = 2 < \infty$

$\therefore \mathrm{Var}\left(\frac{f(x)}{g(x)}\right)$ and $\mathrm{Var}(\hat{I}(f))$ are finite

2° If $g(x) = \frac{3}{2}x^{\frac{1}{2}}$ (please verify on your own that this function is a pdf.)

$\because \int_0^1 \frac{f^2(x)}{g(x)}\,dx = \int_0^1 \frac{2}{3}x^{-1}\,dx = \frac{2}{3}\ln(x)\Big|_0^1 = \infty$

$\therefore \mathrm{Var}\left(\frac{f(x)}{g(x)}\right)$ and $\mathrm{Var}(\hat{I}(f))$ are infinite.

(c)

1° If $g \sim U(0,1)$, then $\hat{I}(f) = \frac{1}{n}\sum_{i=1}^n f(x_i)$

$\mathrm{Var}_U(\hat{I}(f)) = \frac{1}{n}\left[\int_0^1 f^2(x)\,dx - I^2(f)\right]$

(This is the same Monte Carlo estimate as that of Example A of Sec. 5.2.)

2° Let $g^*(x) = \frac{|f(x)|}{\int_0^1 |f(x)|\,dx} \Rightarrow \mathrm{Var}_{g^*}(\hat{I}(f)) = \frac{1}{n}\left[\left(\int_0^1 |f(x)|\,dx\right)^2 - I^2(f)\right]$

By Cauchy Schwarz inequality, for any pdf g (包含 $g \sim U(0,1)$)

$\left(\int_0^1 |f(x)|\,dx\right)^2 \le \left[\int_0^1 \left(\frac{|f(x)|}{\sqrt{g(x)}}\right)^2 dx\right]\left[\int_0^1 \left(\sqrt{g(x)}\right)^2 dx\right]$

$= \int_0^1 \frac{f^2(x)}{g(x)}\,dx \int_0^1 g(x)\,dx = \int_0^1 \frac{f^2(x)}{g(x)}\,dx$ $\because g$ is a pdf.

So $\mathrm{Var}_{g^*}(\hat{I}(f)) = \frac{1}{n}\left[\left(\int_0^1 |f(x)|\,dx\right)^2 - I^2(f)\right] \le \mathrm{Var}_U(\hat{I}(f))$

#23

1° 假設隨機點坐標為 (X,Y)

(X,Y) 的 joint pdf, $f_{X,Y}(x,y) = \begin{cases} 1 & 0\le x\le 1,\ 0\le y\le 1 \\ 0 & \text{o.w.} \end{cases}$

Let $Z = I_A(X,Y) = \begin{cases} 1 & (X,Y)\in \text{object} \\ 0 & \text{o.w.} \end{cases}$

$E(Z) = E(I_A(X,Y)) = P(Z=1) = P((X,Y)\in \text{object}) = \iint_{\text{object}} 1\,dx\,dy = A$ (object area)

and $E(Z^2) = E(Z) = A \Rightarrow \text{Var}(Z) = A - A^2 < \infty$

2° $\because (X_i,Y_i) \xrightarrow{\text{iid}} f_{X,Y}(x,y) = \begin{cases} 1 & 0\le x\le 1,\ 0\le y\le 1 \\ 0 & \text{o.w.} \end{cases}$

Let $Z_i = I_A(X_i,Y_i) = \begin{cases} 1 & (X_i,Y_i)\in \text{object} \\ 0 & \text{o.w.} \end{cases}$ and $Z_1 \cdots Z_n$ are iid with mean A

By WLLN,

$\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow{P} E(Z) = A$ as $n\to\infty$

thus, we can estimate A by $\frac{1}{n}\sum_{i=1}^{n} Z_i$.

#28.

1° $f_n = \begin{cases} \frac{1}{2} & x = (\frac{1}{2})^n \text{ or } -(\frac{1}{2})^n \\ 0 & \text{o.w.} \end{cases}$

① $x=0 \Rightarrow \forall\, n\in\mathbb{N},\ f_n(0)=0 \Rightarrow \lim_{n\to\infty} f_n(0) = 0$

② $x\neq 0 \Rightarrow \because \lim_{n\to\infty} (\frac{1}{2})^n = 0$

$\therefore \exists\, N\in\mathbb{N}$ s.t. for $n>N$, $|(\frac{1}{2})^n| < |x|$

So $\forall\, n>N,\ f_n(x)=0$

thus $\lim_{n\to\infty} f_n(x) = 0$

2° $F_n(x) = \begin{cases} 0 & x < -(\frac{1}{2})^n \\ \frac{1}{2} & -(\frac{1}{2})^n \le x < (\frac{1}{2})^n \\ 1 & x \ge (\frac{1}{2})^n \end{cases}$

① $x<0 \Rightarrow$ as $n\to\infty$, $\exists\, n$ s.t. $-(\frac{1}{2})^n > x \Rightarrow \lim_{n\to\infty} F_n(x) = 0$

② $x>0 \Rightarrow$ as $n\to\infty$ $\exists\, n$ s.t. $(\frac{1}{2})^n < x \Rightarrow \lim_{n\to\infty} F_n(x) = 1$

③ $x=0 \Rightarrow F_n(0) = \frac{1}{2}\ \forall\, n \Rightarrow \lim_{n\to\infty} F_n(x) = \frac{1}{2}$

So $\lim_{n\to\infty} F_n(x) = \begin{cases} 0 & x<0 \\ \frac{1}{2} & x=0 \\ 1 & x>0 \end{cases}$，這不是 cdf，因為在 $x=0$ 不是右連續

Define the cdf $F(x) = \begin{cases} 0, & \text{if } x<0 \\ 1, & \text{if } x\ge 0 \end{cases}$

- for $x<0$: $\lim_{n\to\infty} F_n(x) = 0 = F(x)$
- for $x>0$: $\lim_{n\to\infty} F_n(x) = 1 = F(x)$

→ 因為 converge in distribution 定義要求在 $F(x)$ 的連續點，而 $x=0$ 為不連續點，所以沒有 $x=0$ 仍可成立。

CH 6.

#8.

Let $X \sim \exp(1)$, $Y \sim \exp(1)$

$M_X(t) = M_Y(t) = \frac{1}{1-t}$

Consider $M_{2X}(t) = E(e^{t(2X)}) = \frac{1}{1-2t} \Rightarrow 2X \sim \Gamma(1, \frac{1}{2}) = \chi^2_2$

$\Rightarrow 2X \sim \chi^2_2$, $2Y \sim \chi^2_2$ and $X \perp Y$

$\frac{X}{Y} = \frac{2X/2}{2Y/2} \sim F_{2,2}$

#11.

$X_1 \cdots X_n \overset{iid}{\sim} N(\mu_X, \sigma^2)$, $Y_1 \cdots Y_m \overset{iid}{\sim} N(\mu_Y, \sigma^2)$

Let $\begin{cases} U = \frac{(n-1)S_X^2}{\sigma^2} \sim \chi^2_{n-1} \\ V = \frac{(m-1)S_Y^2}{\sigma^2} \sim \chi^2_{m-1} \end{cases}$, Then U and V are independent,

$F = \frac{U/n-1}{V/m-1} = \frac{S_X^2}{S_Y^2} \sim F(n-1, m-1)$

$P(\frac{S_X^2}{S_Y^2} > 0) = P(F(n-1)(m-1) > 0)$ 查表.