Ch2

21. If X is a geometric random variable, show that

$$P(X > n + k - 1 | X > n - 1) = P(X > k)$$

In light of the construction of a geometric distribution from a sequence of independent Bernoulli trials, how can this be interpreted so that it is "obvious"?

If $X \sim Geo(p)$

$$\Rightarrow P(X>n-1) = \sum_{k=n}^{\infty} P(X=k)$$

$$= \sum_{k=n}^{\infty} p(1-p)^{k-1} = \frac{p(1-p)^{n-1}}{1-(1-p)} = (1-p)^{n-1}$$

$$P(X>n+k-1 | X>n-1) = \frac{P((X>n+k-1)\cap(X>n-1))}{P(X>n-1)} \quad \because k \geq 0$$

$$= \frac{P(X>n+k-1)}{P(X>n-1)}$$

$$= \frac{(1-p)^{n+k-1}}{(1-p)^{n-1}}$$

$$= (1-p)^k$$

$$= P(X>k)$$

If you have failed n-1 times and succeeded n+k-1th times, then you are now in the same situation as if you had started the experiment at the beginning and succeeded at time k. No matter how many times you fail, the next experiment is like starting from scratch.

31. Phone calls are received at a certain residence as a Poisson process with parameter $\lambda = 2$ per hour.

a. If Diane takes a 10-min. shower, what is the probability that the phone rings during that time?

b. How long can her shower be if she wishes the probability of receiving no phone calls to be at most .5?

(a) Define $X \equiv$ The number of calls received per 10 mins at a residence.

$$X \sim Poi(\lambda = 2 \times \frac{1}{6} = \frac{1}{3})$$

$$P(X>0) = 1 - P(X=0) = 1 - \frac{e^{-\frac{1}{3}} \frac{1}{3}^0}{0!} = 1 - e^{-\frac{1}{3}}$$

(b) Define $N(t) \equiv$ Number of calls received every t hours at a residence.

$$N(t) \sim Poi(\lambda = 2t)$$

$$P(N(t)=k) = \frac{(2t)^k e^{-2t}}{k!}$$

$$P(N(t)=0) = e^{-2t} \leq 0.5$$

$$\Leftrightarrow -2t \leq \ln(0.5)$$

$$\Leftrightarrow t \geq \frac{\ln(0.5)}{-2} \approx 0.3466$$

The maximum shower time is 0.3466 hours.

#48 T is an exponential random variable, and $P(T < 1) = .05$. What is λ?

$$T \sim Exp(\lambda) \Rightarrow F(t) = \begin{cases} 1 - e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\begin{aligned} 0.05 &= P(T<1) \\ &= F(1) \\ &= 1 - e^{-\lambda} \end{aligned}$$

$$\Rightarrow e^{-\lambda} = 0.95$$

$$\Rightarrow \lambda = -\ln(0.95) \approx 0.0513$$

61. Find the density of cX when X follows a gamma distribution. Show that only λ is affected by such a transformation, which justifies calling λ a scale parameter.

#61

$X \sim \Gamma(\alpha, \lambda)$

$$\Rightarrow M_X(t) = E(e^{tX}) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}, \quad t < \lambda$$

$$M_{cX}(t) = E(e^{tcX}) = M_X(ct) = \left(\frac{\lambda}{\lambda - ct}\right)^{\alpha} = \left(\frac{\frac{\lambda}{c}}{\frac{\lambda}{c} - t}\right)^{\alpha}, \quad ct < \lambda$$

By uniqueness theorem of moment generating function, $cX \sim \Gamma(\alpha, \frac{\lambda}{c})$

Compare the distributions of X and cX, we note that only λ is affected.

Ch 3

22. Consider a Poisson process on the real line, and denote by $N(t_1, t_2)$ the number of events in the interval (t_1, t_2). If $t_0 < t_1 < t_2$, find the conditional distribution of $N(t_0, t_1)$ given that $N(t_0, t_2) = n$. (*Hint:* Use the fact that the numbers of events in disjoint subsets are independent.)

Claim: $X_1 \sim P(\lambda_1),\ X_2 \sim P(\lambda_2),\ X_1 \perp\!\!\!\perp X_2$

$$\Rightarrow X_1 \mid X_1+X_2=n \sim Bin\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right) \quad \cdots\cdots \langle 1\rangle$$

$\because X_1 \perp\!\!\!\perp X_2$

$$\Rightarrow P(X_1=k, X_2=m) = P(X_1=k)P(X_2=m) = \frac{\lambda_1^k e^{-\lambda_1}}{k!}\frac{\lambda_2^m e^{-\lambda_2}}{m!}$$

$\because X_1 \perp\!\!\!\perp X_2 \quad \therefore X_1+X_2 \sim Poi(\lambda_1+\lambda_2)$ (by LN Ch1~6, P2-68 note)

$$P(X_1+X_2=n) = \frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}$$

$$P(X_1=k, X_1+X_2=n) = P(X_1=k)P(X_2=n-k) = \frac{\lambda_1^k e^{-\lambda_1}}{k!}\frac{\lambda_2^{n-k} e^{-\lambda_2}}{(n-k)!}$$

$$P(X_1=k \mid X_1+X_2=n) = \frac{P(X_1=k, X_1+X_2=n)}{P(X_1+X_2=n)} = \frac{\dfrac{\lambda_1^k e^{-\lambda_1}}{k!}\dfrac{\lambda_2^{n-k} e^{-\lambda_2}}{(n-k)!}}{\dfrac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}}$$

$$= \frac{n!}{k!(n-k)!}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$P(X_1=k \mid X_1+X_2=n) = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$ is pdf of $Bin(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$

Thus $X_1 \mid X_1+X_2=n \sim Bin(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$ #

$N(t_1,t_2) \sim Poi(\lambda(t_2-t_1))$

$N(t_0,t_1) \sim Poi(\lambda(t_1-t_0))$

$\because (t_1,t_2)\cap(t_0,t_1)=\phi \quad \therefore N(t_1,t_2) \perp\!\!\!\perp N(t_0,t_1)$

By <1> we have $N(t_0,t_1) \mid N(t_0,t_2)=n \sim Bin(n, \frac{t_1-t_0}{t_2-t_0})$

#26 Let P have a uniform distribution on $[0, 1]$, and, conditional on $P = p$, let X have a Bernoulli distribution with parameter p. Find the conditional distribution of P given X.

$$P \sim U(0,1), \quad X \mid P=p \sim Ber(p)$$

$$f_{P,X}(p,x) = f_P(p)\, P(X=x \mid P=p)$$
$$= 1 \times p^x (1-p)^{1-x}, \quad p \in (0,1), \quad x = 0 \text{ or } 1$$

$$f_X(x) = \int_0^1 f_{P,X}(p,x)\, dp$$
$$= \int_0^1 p^x (1-p)^{1-x}\, dp$$
$$= \frac{\Gamma(x+1)\Gamma(2-x)}{\Gamma(3)}$$
$$= \frac{x!\,(1-x)!}{2!}$$
$$= \frac{1}{2}, \quad x = 0 \text{ or } 1$$

$$f_{P|X}(p|x) = \frac{f_{P,X}(p,x)}{f_X(x)}$$
$$= \frac{p^x(1-p)^{1-x}}{\frac{1}{2}}$$
$$= 2p^x(1-p)^{1-x}, \quad p \in (0,1)$$

Thus $P \mid X=x \sim Beta(x+1, 2-x)$, for $x = 0$ or 1.

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CH4

~~30.~~ 20. Find $E[1/(X+1)]$, where X is a Poisson random variable.

pf. $$E\left[\frac{1}{X+1}\right] = \sum_{k=0}^{\infty} \frac{1}{k+1} \cdot \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^{k+1} e^{-\lambda}}{(k+1)!} \cdot \frac{1}{\lambda}$$

$$= \frac{1}{\lambda}\left(\underbrace{\sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} + \frac{\lambda^0 e^{-\lambda}}{0!}}_{=1} \right) - \frac{1}{\lambda} \cdot \frac{\lambda^0 e^{-\lambda}}{0!}$$

= 1 because prob. add up to 1

$$= \frac{1}{\lambda} - \frac{1}{\lambda}\left(e^{-\lambda}\right) = \frac{e^{-\lambda}}{\lambda}\left(e^{\lambda}-1\right) \quad \blacksquare$$

42. Let X be an exponential random variable with standard deviation σ. Find $P(|X - E(X)| > k\sigma)$ for $k = 2, 3, 4$, and compare the results to the bounds from Chebyshev's inequality.

pf. $P(|X-E(X)| > k\sigma) = P(X-E(X) < -k\sigma) + P(X-E(X) > k\sigma)$

and since for exponential r.v., $\mu = \sigma$

we have $P(X-E(X) < -k\sigma) = P(X < -(k+1)\sigma) = 0$ for $k=2,3,4$

$$P(|X-E(X)| > k\sigma) = P(X > (1+k)\sigma)$$

$$= e^{-\frac{(1+k)\sigma}{\sigma}} = e^{-(1+k)}$$

using Chebyshev's inequality

$$P(|X-E(X)| > k\sigma) \leq \frac{1}{\lambda^2}$$

	$k=2$	$k=3$	$k=4$
Chebyshev's	$< \frac{1}{4}$	$< \frac{1}{9}$	$< \frac{1}{16}$
Real Value	$e^{-3} \doteq 0.05$	$e^{-4} \doteq 0.018$	$e^{-5} \doteq 0.007$

76. Let the point (X, Y) be uniformly distributed over the half disk $x^2 + y^2 \leq 1$, where $y \geq 0$. If you observe X, what is the best prediction for Y? If you observe Y, what is the best prediction for X? For both questions, "best" means having the minimum mean squared error.

pf. The best guess will be $E_{Y|X}(Y|X)$ (see example 33 in course note)

Since unif. dist. and area of half disk be $\frac{\pi}{2}$,

$f(x,y) = \frac{2}{\pi}$ for $x^2+y^2 \leq 1$ and $y \geq 0$

$$f_X(x) = \int_0^{\sqrt{1-x^2}} \frac{2}{\pi}\, dy = \frac{2}{\pi}\sqrt{1-x^2} \qquad -1 \leq x \leq 1$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi}\, dx = \frac{4}{\pi}\sqrt{1-y^2} \qquad 0 \leq y \leq 1$$

$$E[Y|X] = \int y\, f(y|x)\, dy = \int_0^{\sqrt{1-x^2}} y \cdot \frac{f(x,y)}{f_X(x)}\, dy$$

$$= \frac{\pi}{2\sqrt{1-x^2}} \cdot \frac{2}{\pi} \cdot \left(\frac{1}{2} y^2 \Big|_0^{\sqrt{1-x^2}} \right)$$

$$= \frac{1}{\sqrt{1-x^2}} \cdot \frac{1-x^2}{2} = \frac{\sqrt{1-x^2}}{2}$$

$$E[X|Y] = \int x\, f(x|y)\, dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x \cdot \frac{f(x,y)}{f_Y(y)}\, dx$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x \cdot \frac{2}{\pi} \cdot \frac{\pi}{4\sqrt{1-y^2}}\, dx$$

$$= \frac{1}{2\sqrt{1-y^2}} \left(\frac{1}{2} x^2 \right) \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$$

$$= 0$$

80. Let X be a continuous random variable with density function $f(x) = 2x$, $0 \le x \le 1$. Find the moment-generating function of X, $M(t)$, and verify that $E(X) = M'(0)$ and that $E(X^2) = M''(0)$.

pf.

$$M(t) = E[e^{tx}] = \int_0^1 e^{tx} \cdot 2x \, dx$$

$$= \frac{1}{t}\int_0^1 2x \, de^{tx} = \frac{1}{t}\left(2xe^{tx}\Big|_0^1 - 2\int_0^1 e^{tx} dx \right)$$

$$= \frac{1}{t}\left(2e^t - \frac{2}{t}(e^t - 1) \right)$$

$$= \frac{2te^t - 2e^t + 2}{t^2}$$

$$M'(t) = \frac{2t^2e^t - 4te^t + 4e^t - 4}{t^3}$$

Since $M'(t)$ is $\frac{0}{0}$ when $t=0$, we use L'Hospital

$$\lim_{t\to 0} M'(t) = \lim_{t\to 0} \frac{2t^2e^t - 4te^t + 4e^t - 4}{t^3} \quad \left(\frac{0}{0}\right)$$

$$\overset{LH}{=} \lim_{t\to 0} \frac{4te^t + 2t^2e^t - 4e^t - 4te^t + 4e^t}{3t^2}$$

$$= \lim_{t\to 0} \frac{2t^2e^t}{3t^2} = \frac{2}{3}$$

Same

$$E[x] = \int_0^1 x \cdot 2x \, dx = \frac{2}{3}$$

$$M''(t) = \frac{2t^3 \cdot e^t - 6t^2e^t + 12te^t - 12e^t + 12}{t^4}$$

Since $M''(t)$ is $\frac{0}{0}$ when $t=0$, use L'Hos

$$\lim_{t\to 0} M''(t) = \lim_{t\to 0} \frac{2t^3 \cdot e^t - 6t^2e^t + 12te^t - 12e^t + 12}{t^4} \quad \left(\frac{0}{0}\right)$$

$$\overset{LH}{=} \lim_{t\to 0} \frac{6t^2e^t + 2t^3e^t - 12te^t - 6t^2e^t + 12e^t + 12te^t - 12e^t}{4t^3}$$

$$= \lim_{t\to 0} \frac{2t^3e^t}{4t^3} = \frac{1}{2}$$

$$E[X^2] = \int_0^1 x^2 \cdot 2x \, dx = \frac{2}{4}x^4\Big|_0^1 = \frac{1}{2}$$

same

☑

92. Suppose that Θ is a random variable that follows a gamma distribution with parameters λ and α, where α is an integer, and suppose that, conditional on Θ, X follows a Poisson distribution with parameter Θ. Find the unconditional distribution of $\alpha + X$. (*Hint:* Find the mgf by using iterated conditional expectations.)

pf. $P(X \mid \Theta=\theta) = \frac{\theta^x}{x!} e^{-\theta}$

$$E[e^{tX} \mid \Theta=\theta) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{\theta^x e^{-\theta}}{x!}$$

$$= e^{-\theta} \sum_{x=0}^{\infty} \frac{(e^t\theta)^x}{x!}$$

$$= e^{-\theta} \cdot e^{+(e^t\theta)} \cdot \underbrace{\sum_{x=0}^{\infty} \frac{(e^t\theta)^x e^{-(e^t\theta)}}{x!}}_{=1}$$

$$= e^{\theta(e^t-1)}$$

$$M_X(t) = E[e^{tX}] = E[E[e^{tX} \mid \Theta=\theta]] = E[e^{\theta(e^t-1)}]$$

$$= M_\Theta(e^t-1)$$

$$M_\Theta(t) = \left(\frac{\lambda}{\lambda-t}\right)^\alpha, \ t<\lambda \qquad \text{(gamma dist. mgf.)}$$

$$\Rightarrow M_\Theta(e^t-1) = \left(\frac{\lambda}{\lambda-e^t+1}\right)^\alpha = M_X(t)$$

$$M_{\alpha+X}(t) = e^{t\alpha} \cdot \left(\frac{\lambda}{\lambda-e^t+1}\right)^\alpha = \left(\frac{\lambda e^t}{\lambda-e^t+1}\right)^\alpha$$

$$= \left(\frac{\frac{\lambda}{\lambda+1} e^t}{1 - \frac{e^t}{\lambda+1}} \right)^\alpha$$

By uniqueness thm we have $\alpha + X \sim NB\left(\alpha, \frac{\lambda}{\lambda+1}\right)$

Negative Binomial MGF

$$\left(\frac{pe^t}{1-(1-p)e^t} \right)^r \quad \text{for } t < -\ln(1-p)$$

see p 2-63

so we have $p = \frac{\lambda}{\lambda+1}$ $r = \alpha$

100. If X is uniform on [10, 20], find the approximate and exact mean and variance of $Y = 1/X$, and compare them.

pf.

Exact mean and variance

$$E[Y] = \int_{10}^{20} \frac{1}{x} \cdot \frac{1}{10}\, dx = \frac{1}{10} \ln 2 \doteqdot 0.069$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = \int_{10}^{20} \frac{1}{x^2} \cdot \frac{1}{10}\, dx - \left(\frac{\ln 2}{10}\right)^2$$

$$= \frac{1}{10}\left(-\frac{1}{x}\right)\Big|_{10}^{20} - \frac{(\ln 2)^2}{100}$$

$$= \frac{1}{200} - \frac{(\ln 2)^2}{100} \doteqdot 0.0002$$

Approx. method

$E[X] = 15$ $Var(X) = \frac{10^2}{12} = \frac{25}{3}$

let $g(x) = \frac{1}{x}$ $g'(x) = -\frac{1}{x^2}$ $g''(x) = \frac{2}{x^3}$

by Thm 3.9 (2-56)

$$E\left[\frac{1}{X}\right] \doteqdot g(E(X)) = \frac{1}{15} \doteqdot 0.0\overline{6}$$

$$Var\left(\frac{1}{X}\right) \doteqdot Var(X) \cdot [g'(E[X])]^2$$

$$= \frac{25}{3} \cdot \frac{1}{(15)^4} \doteqdot 0.000165$$

Another more accurate approximate method

$E[g(x)] \doteq g(\mu_x) + \frac{1}{2}\sigma_x^2 g''(\mu_x)$

$= \frac{1}{15} + \frac{1}{2} \cdot \frac{25}{3} \cdot \frac{2}{15^3} \doteq 0.06914$