Chapter 2

Problem 21

Let X be a geometric random variable with success probability p. The probability mass function is given by $P(X = x) = (1-p)^{x-1}p$ for $x = 1, 2, 3, \ldots$.

The probability that $X > m$ (meaning the first m trials are all failures) is:

$$P(X > m) = (1-p)^m$$

To show that $P(X > n+k-1 \mid X > n-1) = P(X > k)$, we use the definition of conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Let A be the event $\{X > n+k-1\}$ and B be the event $\{X > n-1\}$. Since $n+k-1 \geq n-1$, the intersection $A \cap B$ is simply A. Thus:

$$\begin{aligned} P(X > n+k-1 \mid X > n-1) &= \frac{P(X > n+k-1)}{P(X > n-1)} \\ &= \frac{(1-p)^{n+k-1}}{(1-p)^{n-1}} \\ &= (1-p)^{(n+k-1)-(n-1)} \\ &= (1-p)^k \end{aligned}$$

Since $P(X > k) = (1-p)^k$, we have successfully shown that:

$$P(X > n+k-1 \mid X > n-1) = P(X > k)$$

Note. This is called memoryless property.

Problem 30

Let $X \sim \text{Poisson}(\lambda)$, where the probability mass function is $p(k) = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, 2, \ldots$.

Following the hint, we analyze the ratio of consecutive terms $\frac{p(k)}{p(k+1)}$:

Step 1: Calculate the Ratio

$$\begin{aligned} \frac{p(k)}{p(k+1)} &= \frac{\frac{e^{-\lambda}\lambda^k}{k!}}{\frac{e^{-\lambda}\lambda^{k+1}}{(k+1)!}} \\ &= \frac{\lambda^k}{k!} \cdot \frac{(k+1)!}{\lambda^{k+1}} \\ &= \frac{k+1}{\lambda} \end{aligned}$$

Step 2: Find the k_0

We want to find k_0 such that the behavior of the sequence $p(k)$ changes:

1. **Increasing phase:** $p(k)$ is increasing when $\frac{p(k)}{p(k+1)} < 1$.

$$\frac{k+1}{\lambda} < 1 \implies k+1 < \lambda \implies k < \lambda - 1$$

2. **Decreasing phase:** $p(k)$ is non-increasing when $\frac{p(k)}{p(k+1)} \geq 1$.

$$\frac{k+1}{\lambda} \geq 1 \implies k+1 \geq \lambda \implies k \geq \lambda - 1$$

Thus, the threshold k_0 is the smallest integer such that $k_0 \geq \lambda - 1$ (i.e., $\lceil \lambda - 1 \rceil$) or the greatest integer such that $k_0 \leq \lambda$ (i.e., $\lfloor \lambda \rfloor$). When λ is not an integer, $\lceil \lambda - 1 \rceil = \lfloor \lambda \rfloor$.

Conclusion

The maximum value of $p(k)$ occurs at the transition point:

- If λ is **not an integer**: The function increases until $k < \lambda - 1$ and starts decreasing for $k > \lambda - 1$. The maximum occurs at $k = \lfloor \lambda \rfloor = \lceil \lambda - 1 \rceil$.
- If λ is **an integer**: When $k = \lambda - 1$, the ratio $\frac{p(k)}{p(k+1)} = \frac{(\lambda-1)+1}{\lambda} = 1$, which means $p(\lambda - 1) = p(\lambda)$. In this case, there are two modes at $k = \lambda - 1$ and $k = \lambda$.

Problem 44

Let T be a continuous random variable following an exponential distribution with parameter λ. Its probability density function (PDF) is:

$$f_T(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$

And its cumulative distribution function (CDF) is:

$$F_T(t) = P(T \leq t) = 1 - e^{-\lambda t}, \quad t \geq 0$$

We define a discrete random variable X such that $X = k$ if $k \leq T < k+1$ for $k = 0, 1, 2, \ldots$.

Step 1: Find the Frequency Function (PMF) of X

The frequency function $P(X = k)$ is the probability that T falls in the interval $[k, k+1)$:

$$\begin{aligned} P(X = k) &= P(k \leq T < k+1) \\ &= F_T(k+1) - F_T(k) \\ &= (1 - e^{-\lambda(k+1)}) - (1 - e^{-\lambda k}) \\ &= e^{-\lambda k} - e^{-\lambda(k+1)} \\ &= e^{-\lambda k} - e^{-\lambda k} \cdot e^{-\lambda} \\ &= e^{-\lambda k}(1 - e^{-\lambda}) \end{aligned}$$

Step 2: Identify the Distribution

Let $p = 1 - e^{-\lambda}$. Then $1 - p = e^{-\lambda}$. Substituting these into the expression:

$$P(X = k) = (1 - p)^k p, \quad k = 0, 1, 2, \ldots$$

Conclusion

The frequency function of X is $P(X = k) = e^{-\lambda k}(1 - e^{-\lambda})$. This shows that X follows a **Geometric distribution** (specifically, the version that counts the number of failures before the first success), where the "success" probability is $p = 1 - e^{-\lambda}$.

Problem 55

Given that X follows a normal distribution with mean μ and variance σ^2, denoted as $X \sim N(\mu, \sigma^2)$. We want to find the value of c in terms of σ such that:

$$P(\mu - c \leq X \leq \mu + c) = 0.95$$

Step 1: Standardization

To solve this, we convert X to the standard normal random variable $Z \sim N(0, 1)$ using the transformation $Z = \frac{X-\mu}{\sigma}$.

Subtracting μ and dividing by σ for all terms inside the probability:

$$P\left(\frac{(\mu - c) - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{(\mu + c) - \mu}{\sigma}\right) = 0.95$$
$$P\left(-\frac{c}{\sigma} \leq Z \leq \frac{c}{\sigma}\right) = 0.95$$

Step 2: Using the Standard Normal Table

Since the standard normal distribution is symmetric about 0, the area in the middle is 0.95, which means the total area in the two tails is:

$$1 - 0.95 = 0.05$$

Each tail (left and right) contains an area of $0.05/2 = 0.025$.

Therefore, we are looking for a value $z = \frac{c}{\sigma}$ such that $P(Z \leq z) = 1 - 0.025 = 0.975$.

Referring to **Table 2 in Appendix B** (Standard Normal Probabilities): Looking for the value 0.9750 in the body of the table, we find the corresponding z-score is:

$$z = 1.96$$

Step 3: Solve for c

Setting our standardized value equal to the z-score:

$$\frac{c}{\sigma} = 1.96$$
$$c = 1.96\sigma$$

Conclusion

The value of c such that $P(\mu - c \leq X \leq \mu + c) = 0.95$ is approximately 1.96σ.

Problem 61

Let X follow a Gamma distribution with shape parameter α and rate parameter λ, denoted as $X \sim \text{Gamma}(\alpha, \lambda)$. As suggested in the hint, we will use the **Moment Generating Function (MGF)** to find the distribution of $Y = cX$, where $c > 0$.

Step 1: The MGF of X

The MGF of a Gamma random variable X is given by:

$$M_X(t) = E[e^{tX}] = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}, \quad \text{for } t < \lambda$$

Step 2: Find the MGF of $Y = cX$

By the definition of MGF for a transformed variable:

$$M_Y(t) = E[e^{tY}] = E[e^{t(cX)}] = E[e^{(ct)X}]$$

This is equivalent to evaluating the MGF of X at the point ct:

$$\begin{aligned} M_Y(t) &= M_X(ct) \\ &= \left(\frac{\lambda}{\lambda - ct}\right)^{\alpha} \end{aligned}$$

Step 3: Simplify the Expression

To identify the distribution, we rearrange the terms so the denominator is in the form (new λ) $- t$:

$$M_Y(t) = \left(\frac{\lambda/c}{\lambda/c - t}\right)^{\alpha}$$

Step 4: Identify the Resulting Distribution

Notice that this is exactly the form of a MGF for a Gamma distribution with:

- New shape parameter: $\alpha' = \alpha$
- New rate parameter: $\lambda' = \frac{\lambda}{c}$

Thus, $Y = cX \sim \text{Gamma}(\alpha, \lambda/c)$.

Chapter 3

Problem 22

Consider a Poisson process with intensity $\lambda > 0$. Let:

- $X = N(t_0, t_1)$, the number of events in (t_0, t_1).
- $Y = N(t_1, t_2)$, the number of events in (t_1, t_2).

Since the intervals (t_0, t_1) and (t_1, t_2) are disjoint, X and Y are independent Poisson random variables with parameters:

$$\lambda_1 = \lambda(t_1 - t_0)$$
$$\lambda_2 = \lambda(t_2 - t_1)$$

The total number of events in the entire interval (t_0, t_2) is $S = X + Y = N(t_0, t_2)$, which follows a Poisson distribution with parameter $\lambda_1 + \lambda_2 = \lambda(t_2 - t_0)$.

Step 1: Calculate the Conditional Probability

We want to find $P(X = k \mid X + Y = n)$ for $k = 0, 1, \ldots, n$:

$$\begin{aligned} P(X = k \mid X + Y = n) &= \frac{P(X = k, X + Y = n)}{P(X + Y = n)} \\ &= \frac{P(X = k, Y = n - k)}{P(X + Y = n)} \end{aligned}$$

Using the independence of X and Y:

$$\begin{aligned} \frac{P(X = k)P(Y = n - k)}{P(X + Y = n)} &= \frac{\left(\frac{e^{-\lambda_1}\lambda_1^k}{k!}\right)\left(\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\right)}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}} \\ &= \frac{n!}{k!(n-k)!} \cdot \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n} \\ &= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k} \end{aligned}$$

Step 2: Simplify the Parameters

Substitute λ_1 and λ_2 back into the success probability p:

$$p = \frac{\lambda(t_1 - t_0)}{\lambda(t_1 - t_0) + \lambda(t_2 - t_1)} = \frac{t_1 - t_0}{t_2 - t_0}$$

Conclusion

The conditional distribution of $N(t_0, t_1)$ given $N(t_0, t_2) = n$ is a **Binomial distribution** with parameters n and $p = \frac{t_1 - t_0}{t_2 - t_0}$.

$$N(t_0, t_1) \mid N(t_0, t_2) = n \sim \text{Binomial}\left(n, \frac{t_1 - t_0}{t_2 - t_0}\right)$$

Solution Manual **Homework 2**

Email: didil0907@gmail.com (李友棣); stat_hou@gapp.nthu.edu.tw(侯秉逸)

NTHU MATH 2820, Statistics (Feb ∼ Jun 2026) Term: Spring 2026

Instructor: Shao-Wei Cheng Date: March 15, 2026

Problem 3.47.
Let X and Y be independent standard normal random variables. Find the density of $Z = X + Y$, and show that Z is normally distributed as well. (Hint: Use the technique of completing the square to help in evaluating the integral.)

Solution: Since X and Y are independent standard normal random variables, their probability density functions are:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

The density of the sum $Z = X + Y$ is given by the convolution of f_X and f_Y:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-x^2/2} e^{-(z-x)^2/2}\, dx$$

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[x^2 + (z^2 - 2zx + x^2)]}\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(2x^2 - 2zx + z^2)}\, dx$$

Now, we complete the square in the exponent for x:

$$2x^2 - 2zx + z^2 = 2\left(x^2 - zx + \frac{z^2}{4}\right) + \frac{z^2}{2} = 2\left(x - \frac{z}{2}\right)^2 + \frac{z^2}{2}$$

Substitute this back into the integral:

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(x - \frac{z}{2}\right)^2 - \frac{z^2}{4}}\, dx = \frac{1}{2\pi} e^{-z^2/4} \int_{-\infty}^{\infty} e^{-\left(x - \frac{z}{2}\right)^2}\, dx$$

Let $u = x - \frac{z}{2}$, then $du = dx$:

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$$

Thus, the density of Z is:

$$f_Z(z) = \frac{1}{2\pi} e^{-z^2/4} \sqrt{\pi} = \frac{1}{\sqrt{4\pi}} e^{-z^2/4} = \frac{1}{\sqrt{2\pi(2)}} e^{-\frac{z^2}{2(2)}}$$

This is the probability density function of a normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 2$. Therefore, $Z \sim N(0, 2)$, which shows that Z is normally distributed.

Problem 4.20.
Find $E[1/(X+1)]$, where X is a Poisson random variable. [Hint: try sum-to-one method]

Solution: For a Poisson random variable X with parameter λ, the probability mass function is $P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}$ for $x=0,1,2,\ldots$ The expected value is:

$$E\left[\frac{1}{X+1}\right]=\sum_{x=0}^{\infty}\frac{1}{x+1}\frac{e^{-\lambda}\lambda^x}{x!}=\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{(x+1)!}$$

To use the sum-to-one method (making it look like a Poisson pmf), we multiply and divide the expression by λ:

$$E\left[\frac{1}{X+1}\right]=\frac{1}{\lambda}\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}$$

Let $y=x+1$. When x goes from 0 to ∞, y goes from 1 to ∞:

$$E\left[\frac{1}{X+1}\right]=\frac{1}{\lambda}\sum_{y=1}^{\infty}\frac{e^{-\lambda}\lambda^{y}}{y!}$$

We know that the sum of the Poisson pmf over all possible values is 1: $\sum_{y=0}^{\infty}\frac{e^{-\lambda}\lambda^y}{y!}=1$. Therefore, the sum from $y=1$ to ∞ is the total sum minus the $y=0$ term:

$$\sum_{y=1}^{\infty}\frac{e^{-\lambda}\lambda^y}{y!}=1-P(Y=0)=1-\frac{e^{-\lambda}\lambda^0}{0!}=1-e^{-\lambda}$$

Substitute this back:

$$E\left[\frac{1}{X+1}\right]=\frac{1}{\lambda}(1-e^{-\lambda})$$

Problem 4.42.
Let X be an exponential random variable with standard deviation σ. Find $P(|X - E(X)| > k\sigma)$ for $k = 2, 3, 4$, and compare the results to the bounds from Chebyshev's inequality. [Hint: Use the cdf, mean, variance equations given in LNp.71 to find the probability]

Solution: Let $X \sim \text{Exp}(\lambda)$. The mean is $E(X) = \frac{1}{\lambda}$ and the variance is $\text{Var}(X) = \frac{1}{\lambda^2}$. Thus, the standard deviation is $\sigma = \frac{1}{\lambda}$. Note that $E(X) = \sigma$. The cumulative distribution function (CDF) is $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$.

We want to find $P(|X - \sigma| > k\sigma)$.

$$|X - \sigma| > k\sigma \iff X - \sigma > k\sigma \quad \text{or} \quad X - \sigma < -k\sigma$$

$$X > (k+1)\sigma \quad \text{or} \quad X < (1-k)\sigma$$

Since X is an exponential random variable, $X \geq 0$. For $k \geq 2$, $(1-k)\sigma < 0$, so $P(X < (1-k)\sigma) = 0$. Thus, the probability is simply:

$$P(X > (k+1)\sigma) = 1 - F((k+1)\sigma) = e^{-\lambda(k+1)\sigma}$$

Since $\sigma = 1/\lambda$, we have $\lambda\sigma = 1$:

$$P(|X - E(X)| > k\sigma) = e^{-(k+1)}$$

Now we compare this to Chebyshev's inequality, which states $P(|X - E(X)| \geq k\sigma) \leq \frac{1}{k^2}$.

- **For $k = 2$:** Exact Probability: $e^{-(2+1)} = e^{-3} \approx 0.0498$ Chebyshev's Bound: $\leq \frac{1}{2^2} = 0.25$
- **For $k = 3$:** Exact Probability: $e^{-(3+1)} = e^{-4} \approx 0.0183$ Chebyshev's Bound: $\leq \frac{1}{3^2} \approx 0.1111$
- **For $k = 4$:** Exact Probability: $e^{-(4+1)} = e^{-5} \approx 0.0067$ Chebyshev's Bound: $\leq \frac{1}{4^2} = 0.0625$

In all cases, the exact probability is much strictly smaller than the upper bound provided by Chebyshev's inequality.

Definition 4.10 (Exponential distribution $E(\lambda)$, sec 2.2.1)

- **pdf:** $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$
- **cdf:** $F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$
- **mgf:** $\frac{\lambda}{\lambda - t}, \quad t < \lambda$
- **mean:** $\frac{1}{\lambda}$
- **variance:** $\frac{1}{\lambda^2}$
- **parameter:** $\lambda > 0$
- **example:** lifetime or waiting time

Problem 4.79.
Let X be a discrete random variable that takes on values $0, 1, 2$ with probabilities $\frac{1}{2}, \frac{3}{8}, \frac{1}{8}$, respectively. Find the moment-generating function of X, $M(t)$, and verify that $E(X) = M'(0)$ and that $E(X^2) = M''(0)$.

Solution: The moment-generating function (MGF) is defined as $M(t) = E[e^{tX}]$. For a discrete random variable, this is:

$$M(t) = \sum_x e^{tx} P(X = x)$$

Substituting the given values:

$$M(t) = e^{t(0)}\left(\frac{1}{2}\right) + e^{t(1)}\left(\frac{3}{8}\right) + e^{t(2)}\left(\frac{1}{8}\right) = \frac{1}{2} + \frac{3}{8}e^t + \frac{1}{8}e^{2t}$$

First, let's calculate the expected values directly from the definition to use for verification later:

$$E(X) = \sum xP(X = x) = 0\left(\frac{1}{2}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{1}{8}\right) = \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$$

$$E(X^2) = \sum x^2 P(X = x) = 0^2\left(\frac{1}{2}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{1}{8}\right) = \frac{3}{8} + \frac{4}{8} = \frac{7}{8}$$

Now, let's find the derivatives of $M(t)$:

$$M'(t) = \frac{d}{dt}\left(\frac{1}{2} + \frac{3}{8}e^t + \frac{1}{8}e^{2t}\right) = \frac{3}{8}e^t + \frac{2}{8}e^{2t} = \frac{3}{8}e^t + \frac{1}{4}e^{2t}$$

$$M''(t) = \frac{d}{dt}\left(\frac{3}{8}e^t + \frac{1}{4}e^{2t}\right) = \frac{3}{8}e^t + \frac{2}{4}e^{2t} = \frac{3}{8}e^t + \frac{1}{2}e^{2t}$$

Finally, evaluate the derivatives at $t = 0$:

$$M'(0) = \frac{3}{8}e^0 + \frac{1}{4}e^0 = \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$$

$$M''(0) = \frac{3}{8}e^0 + \frac{1}{2}e^0 = \frac{3}{8} + \frac{4}{8} = \frac{7}{8}$$

Thus, we have verified that $E(X) = M'(0) = \frac{5}{8}$ and $E(X^2) = M''(0) = \frac{7}{8}$.

Problem 4.89.
Let $X_1, X_2, \ldots, X_n$ be independent normal random variables with means μ_i and variances σ_i^2. Show that $Y = \sum_{i=1}^n \alpha_i X_i$, where the α_i are scalars, is normally distributed, and find its mean and variance. (Hint: Use moment-generating functions.) (For practice purpose, you must use moment generating function to show it and to find its mean and variance. Other solutions are not acceptable.)

Solution: The moment-generating function of a normal random variable $X_i \sim N(\mu_i, \sigma_i^2)$ is known to be:

$$M_{X_i}(t) = \exp\left(\mu_i t + \frac{1}{2}\sigma_i^2 t^2\right)$$

First, we find the MGF of $\alpha_i X_i$:

$$M_{\alpha_i X_i}(t) = E[e^{t(\alpha_i X_i)}] = E[e^{(\alpha_i t)X_i}] = M_{X_i}(\alpha_i t)$$

$$M_{\alpha_i X_i}(t) = \exp\left(\mu_i(\alpha_i t) + \frac{1}{2}\sigma_i^2(\alpha_i t)^2\right) = \exp\left((\alpha_i\mu_i)t + \frac{1}{2}(\alpha_i^2\sigma_i^2)t^2\right)$$

Since $X_1, X_2, \ldots, X_n$ are independent, the random variables $\alpha_1 X_1, \alpha_2 X_2, \ldots, \alpha_n X_n$ are also independent. The MGF of the sum of independent random variables is the product of their individual MGFs. Thus, the MGF of $Y = \sum_{i=1}^n \alpha_i X_i$ is:

$$M_Y(t) = \prod_{i=1}^n M_{\alpha_i X_i}(t) = \prod_{i=1}^n \exp\left((\alpha_i\mu_i)t + \frac{1}{2}(\alpha_i^2\sigma_i^2)t^2\right)$$

$$= \exp\left(\sum_{i=1}^n \left[(\alpha_i\mu_i)t + \frac{1}{2}(\alpha_i^2\sigma_i^2)t^2\right]\right)$$

$$= \exp\left(\left(\sum_{i=1}^n \alpha_i\mu_i\right)t + \frac{1}{2}\left(\sum_{i=1}^n \alpha_i^2\sigma_i^2\right)t^2\right)$$

This expression takes the exact form of the moment-generating function of a normal distribution $\exp(\mu_Y t + \frac{1}{2}\sigma_Y^2 t^2)$. Because the MGF uniquely determines the distribution, we conclude that Y is normally distributed with:

$$\text{Mean: } \mu_Y = E(Y) = \sum_{i=1}^n \alpha_i\mu_i$$

$$\text{Variance: } \sigma_Y^2 = \text{Var}(Y) = \sum_{i=1}^n \alpha_i^2\sigma_i^2$$

Problem 4.99.
Find expressions for the approximate mean and variance of $Y = g(X)$ for (a) $g(x) = \sqrt{x}$, (b) $g(x) = \log x$, and (c) $g(x) = \sin^{-1} x$. [Hint: use δ method]

Solution: Using the Delta method (Taylor series expansion around $\mu = E(X)$), the approximations for the mean and variance of $Y = g(X)$ are:

$$E[g(X)] \approx g(\mu) \quad \text{(First-order approximation)}$$

$$E[g(X)] \approx g(\mu) + \frac{g''(\mu)}{2}\sigma^2 \quad \text{(Second-order approximation)}$$

$$\text{Var}[g(X)] \approx [g'(\mu)]^2\sigma^2 \quad \text{(First-order approximation)}$$

where $\mu = E(X)$ and $\sigma^2 = \text{Var}(X)$.

(a) $g(x) = \sqrt{x} = x^{1/2}$

$$g'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$g''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x^{3/2}}$$

Mean: $E(Y) \approx \sqrt{\mu}$ (1st-order) or $E(Y) \approx \sqrt{\mu} - \frac{\sigma^2}{8\mu^{3/2}}$ (2nd-order)

Variance: $\text{Var}(Y) \approx \left(\frac{1}{2\sqrt{\mu}}\right)^2 \sigma^2 = \frac{\sigma^2}{4\mu}$

(b) $g(x) = \log x$

$$g'(x) = \frac{1}{x}$$

$$g''(x) = -\frac{1}{x^2}$$

Mean: $E(Y) \approx \log \mu$ (1st-order) or $E(Y) \approx \log \mu - \frac{\sigma^2}{2\mu^2}$ (2nd-order)

Variance: $\text{Var}(Y) \approx \left(\frac{1}{\mu}\right)^2 \sigma^2 = \frac{\sigma^2}{\mu^2}$

(c) $g(x) = \sin^{-1} x$

$$g'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$$

$$g''(x) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x) = \frac{x}{(1-x^2)^{3/2}}$$

Mean: $E(Y) \approx \sin^{-1} \mu$ (1st-order) or $E(Y) \approx \sin^{-1} \mu + \frac{\mu\sigma^2}{2(1-\mu^2)^{3/2}}$ (2nd-order)

Variance: $\text{Var}(Y) \approx \left(\frac{1}{\sqrt{1-\mu^2}}\right)^2 \sigma^2 = \frac{\sigma^2}{1-\mu^2}$