Ch2

#40 $f(x)=\begin{cases} cx^2 & 0\le x\le 1 \\ 0 & \text{o.w} \end{cases}$

(a) $\int_{-\infty}^{\infty} f(x)dx=\int_0^1 cx^2dx=1 \Rightarrow \int_0^1 cx^2dx=\frac{c}{3}x^3\Big|_0^1=\frac{c}{3}=1\Rightarrow c=3$

(b) $F(x)=\int_0^x 3t^2dt=x^3 \quad 0<x<1$

$F(x)=\begin{cases} 1 & x\ge 1 \\ x^3 & 0\le x<1 \\ 0 & x<0 \end{cases}$

(c) $P(0.1\le X<0.5)=\int_{0.1}^{0.5} 3x^2dx=x^3\Big|_{0.1}^{0.5}=0.124=F(0.5)-F(0.1)$

#70

Suppose R is the radius of a circle

$R\sim exp(\lambda)\Rightarrow f_R(r)=\begin{cases} \lambda e^{-\lambda r} & r>0 \\ 0 & \text{o.w} \end{cases}$

Let A be the area of the circle.

$A=\pi R^2\Rightarrow r=(\frac{a}{\pi})^{\frac{1}{2}}.\ |J|=|\frac{dr}{da}|=\frac{1}{2}(\frac{a}{\pi})^{-\frac{1}{2}}\cdot\frac{1}{\pi}=\frac{1}{2}(\pi a)^{-\frac{1}{2}}$

$f_A(a)=f_R((\frac{a}{\pi})^{\frac{1}{2}})|J|=\lambda e^{-\lambda(\frac{a}{\pi})^{\frac{1}{2}}}\frac{1}{2}(\pi a)^{-\frac{1}{2}}=\begin{cases} \frac{\lambda}{2\sqrt{\pi}}a^{-\frac{1}{2}}e^{-\lambda(\frac{a}{\pi})^{\frac{1}{2}}} & a>0 \\ 0 & \text{o.w} \end{cases}$

Ch3

#10

(a) $f_x(x)=\int_0^\infty f(x.y)dy=\int_0^\infty xe^{-x(y+1)}dy=xe^{-x}\int_0^\infty e^{-xy}dy=xe^{-x}\cdot(\frac{-1}{x}e^{-xy}\Big|_0^\infty)$

$=\begin{cases} e^{-x} & 0<x<\infty \\ 0 & \text{o.w} \end{cases}$

$f_Y(y)=\int_0^\infty f(x.y)dx=\int_0^\infty xe^{-x(y+1)}dx=-\frac{1}{y+1}xe^{-x(y+1)}\Big|_0^\infty+\int_0^\infty\frac{1}{y+1}e^{-x(y+1)}dx$

$=-(\frac{1}{y+1})^2e^{-x(y+1)}\Big|_0^\infty=\begin{cases} \frac{1}{(y+1)^2} & 0<y<\infty \\ 0 & \text{o.w} \end{cases}$

$f(x.y)\neq f(x)f(y)\Rightarrow X$ and Y are not independent.

#10

(b) $f(x|y)=\frac{f(x,y)}{f(y)}=\frac{xe^{-x(y+1)}}{\frac{1}{(y+1)^2}}=\begin{cases}x(y+1)^2e^{-x(y+1)} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{O.W}\end{cases}$

$f(y|x)=\frac{f(x,y)}{f(x)}=\frac{xe^{-x(y+1)}}{e^{-x}}=\begin{cases}xe^{-xy} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{O.W}\end{cases}$

#21

令 $I=\begin{cases}1 & \text{if } X \text{ is detected} \\ 0 & \text{O.W}\end{cases}$，則 $P(I=1|X=x)=R(x)$

令 $G_1(y)=P(Y\le y)=P(X\le y|I=1)=\frac{P(X\le y, I=1)}{P(I=1)}=\frac{\int_0^y f_{x,I}(x,1)dx}{\int_0^\infty f_{x,I}(x,1)dx}$

其中 $f_{x,I}$ 為 (X,I) 的 joint pdf/pmf

因為 $f_{x,I}(x,1)=f_x(x)\cdot P(I=1|X=x)=f_x(x)\cdot R(x)\Rightarrow G_1(y)=\frac{\int_0^y f_x(x)R(x)dx}{\int_0^\infty f_x(x)R(x)dx}$

$\Rightarrow g(y)=\frac{d}{dy}G_1(y)=\frac{R(y)f(y)}{\int_0^\infty R(x)f(x)dx}$

#25.

Suppose $S=\pm1$ where $P(S=1)=\frac{1}{2}$, $P(S=-1)=\frac{1}{2}$

S and X are independent $\Rightarrow f_{X,S}(x,s)=f_X(x)f_S(s)$

Let $Y=SX$, $Z=S\Rightarrow x=\frac{y}{z}$, $s=z$, $|J|=1\begin{vmatrix}\frac{1}{z} & 0\\ \frac{-y}{z^2} & 1\end{vmatrix}=\left|\frac{1}{z}\right|$

$f_{Y,Z}(y,z)=f_{X,S}(\frac{y}{z},z)|J|=|\frac{1}{z}|f_X(\frac{y}{z})f(z)$

$f_Y(y)=\sum_{z=1,-1}|\frac{1}{z}|f_X(\frac{y}{z})f(z)=|\frac{1}{1}|f_X(y)\cdot\frac{1}{2}+|\frac{1}{-1}|f_X(-y)\cdot\frac{1}{2}=\frac{1}{2}(f_X(y)+f_X(-y))$

$\Rightarrow f_Y(y)=f_Y(-y)$

#53

$U_1, U_2, U_3 \overset{iid}{\sim} U(0,1) \Rightarrow f_{U_1,U_2,U_3}(u_1,u_2,u_3) = f_{U_1}(u_1) f_{U_2}(u_2) f_{U_3}(u_3) = 1,\ 0<u_1,u_2,u_3<1$
$= 0$ o.w.

We want $U_3^2 > U_1 U_2 \Leftrightarrow \frac{U_1 U_2}{U_3^2} < 1$

$X = \frac{U_1 U_2}{U_3^2},\ Y = U_1,\ Z = U_2$

$\Rightarrow u_1 = y,\ u_2 = z,\ u_3 = \left(\frac{yz}{x}\right)^{\frac{1}{2}},\ |J| = \left| \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2}\left(\frac{yz}{x^3}\right)^{\frac{1}{2}} & \frac{1}{2}\left(\frac{z}{yx}\right)^{\frac{1}{2}} & \frac{1}{2}\left(\frac{y}{zx}\right)^{\frac{1}{2}} \end{matrix} \right| = \frac{1}{2}\left(\frac{yz}{x^3}\right)^{\frac{1}{2}}$

$f_{X,Y,Z}(x,y,z) = f_{U_1,U_2,U_3}\left(y, z, \left(\frac{yz}{x}\right)^{\frac{1}{2}}\right)|J| = \frac{1}{2}\left(\frac{yz}{x^3}\right)^{\frac{1}{2}},\ 0<y<1,\ 0<z<1,\ yz<x<\infty$
$= 0$ o.w.

$$P(X<1) = \int_0^1\int_0^1\int_{yz}^1 \frac{1}{2}\left(\frac{yz}{x^3}\right)^{\frac{1}{2}} dx\,dy\,dz = \int_0^1\int_0^1 \left(-\left(\frac{yz}{x}\right)^{\frac{1}{2}}\Big|_{yz}^1\right) dy\,dz = \int_0^1\int_0^1 (1-\sqrt{yz})\,dy\,dz$$

$$= \int_0^1 \left(y - \frac{2}{3}\sqrt{y^3 z}\right)\Big|_0^1 dz = \int_0^1 \left(1 - \frac{2}{3}\sqrt{z}\right) dz = \left(z - \frac{4}{9}\sqrt{z^3}\right)\Big|_0^1 = \frac{5}{9}$$

Ch3

M. $U_1, \cdots, U_n \overset{iid}{\sim} U(0,1)$

$$f_{U_{(k-1)}, U_{(k)}}(x,y) = \begin{cases} \frac{n!}{(k-2)!(n-k)!} x^{k-2} \cdot (1-y)^{n-k}, & 0 \le x \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

設 $W = U_{(k)} - U_{(k-1)}$, $Z = U_{(k-1)}$ $\Rightarrow$ $U_{(k-1)} = Z$, $U_{(k)} = W + Z$

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

$$f_{W,Z}(w,z) = f_{U_{(k-1)}, U_{(k)}}(z, w+z) \cdot |J|$$

$$= \frac{n!}{(k-2)!(n-k)!} z^{k-2} (1-w-z)^{n-k}, \quad 0 < z < w+z < 1$$

$$\Leftrightarrow 0 < z < 1-w$$

$$f_W(w) = \int_0^{1-w} \frac{n!}{(k-2)!(n-k)!} z^{k-2} (1-z-w)^{n-k} dz$$

(let $u = \frac{z}{1-w}$, $du = \frac{dz}{1-w}$, $dz = (1-w)du$)

$$= \int_0^1 \frac{n!}{(k-2)!(n-k)!} ((1-w)u)^{k-2} (1-(1-w)u-w)^{n-k} (1-w) du$$

$$= \int_0^1 \frac{n!}{(k-2)!(n-k)!} ((1-w)u)^{k-2} ((1-u)(1-w))^{n-k} (1-w) du$$

$$= \frac{n!}{(k-2)!(n-k)!} (1-w)^{n-1} \int_0^1 u^{k-2} (1-u)^{n-k} du$$

$$= \frac{n!}{(k-2)!(n-k)!} (1-w)^{n-1} \frac{\Gamma(k-1)\Gamma(n-k+1)}{\Gamma(n)}$$

$$= \frac{n!}{\cancel{(k-2)!(n-k)!}} (1-w)^{n-1} \frac{\cancel{(k-2)!(n-k)!}}{(n-1)!}$$

$$= n(1-w)^{n-1}, \quad 0 < w < 1$$

Ch4

26.

設 X 為分裂的位置 $X \sim U(0,1)$

$$Y = \max(X, 1-X) = \begin{cases} X & \frac{1}{2} \le X < 1 \\ 1-X & 0 < X < \frac{1}{2} \end{cases}$$

$$F_Y(y) = P(Y \le y) = P(X \le y, \frac{1}{2} \le X < 1) + P(1-X \le y, 0 < X < \frac{1}{2})$$

$$= P(\frac{1}{2} \le X \le y) + P(1-y \le X < \frac{1}{2})$$

$$= y - \frac{1}{2} + \frac{1}{2} - (1-y) = 2y - 1 \qquad \frac{1}{2} < y < 1$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) \Rightarrow f_Y(y) = \begin{cases} 2, & \frac{1}{2} < y < 1 \\ 0 & \text{o.w.} \end{cases} \qquad \therefore Y \sim U(\frac{1}{2}, 1)$$

$$E(Y) = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$$

∴ the expected ratio of the length of the longer piece to the length of the shorter piece is $\frac{E(Y)}{E(1-Y)} = \frac{3/4}{1/4} = 3$

另一個可接受的答案是 $E(\frac{Y}{1-Y}) = \int_{\frac{1}{2}}^{1} \frac{y}{1-y} \times 2 \, dy$

$$= 2\int_{\frac{1}{2}}^{1} (-1 + \frac{1}{1-y}) dy$$

$$= 2[-y|_{\frac{1}{2}}^{1} - \ln(1-y)|_{\frac{1}{2}}^{1}]$$

$$= 2[-\frac{1}{2} - \ln(0) + \ln\frac{1}{2}] = +\infty$$

∴ expectation does not exist

49.

(a) $E(Z) = E(\alpha X + (1-\alpha)Y)$

$= \alpha E(X) + (1-\alpha)E(Y)$

$= \alpha\mu + (1-\alpha)\mu = \mu$

(b) $Var(Z) = Var(\alpha X + (1-\alpha)Y)$

$= \alpha^2 Var(X) + (1-\alpha)^2 Var(Y)$

$= \alpha^2\sigma_X^2 + (1-\alpha)^2\sigma_Y^2 \equiv f(\alpha)$

$\frac{\partial}{\partial\alpha} f(\alpha) = 2\alpha\sigma_X^2 - 2(1-\alpha)\sigma_Y^2 = 0$

$\alpha\sigma_X^2 = (1-\alpha)\sigma_Y^2$

$\alpha(\sigma_X^2 + \sigma_Y^2) = \sigma_Y^2$

$\alpha = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$ check $\left.\frac{\partial^2}{\partial\alpha^2} f(\alpha)\right|_{\alpha} > 0$

(c) $Z = \frac{X+Y}{2}$ when $\alpha = \frac{1}{2}$

$Var(Z) = \frac{1}{4}(\sigma_X^2 + \sigma_Y^2)$

The average doesn't affect the location of the mean, so the considerations of better rest solely on variance considerations.

Because $Var(Z) < Var(X) \Leftrightarrow \frac{\sigma_X^2+\sigma_Y^2}{4} < \sigma_X^2 \Leftrightarrow \frac{\sigma_X^2}{\sigma_Y^2} > 1/3$

$Var(Z) < Var(Y) \Leftrightarrow \frac{\sigma_X^2+\sigma_Y^2}{4} < \sigma_Y^2 \Leftrightarrow \frac{\sigma_X^2}{\sigma_Y^2} < 3$

when $\frac{1}{3} < \frac{\sigma_X^2}{\sigma_Y^2} < 3$. Z would be better (having small variance) than X or Y alone. Otherwise the variance of Z will exceed variance of X or Y

59.

$$f(x,y)=\frac{1}{\pi} \quad 0<x^2+y^2<1$$

$$f_Y(y)=\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\,dx=\frac{2\sqrt{1-y^2}}{\pi} \quad -1<y<1$$

$$f_X(x)=\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy=\frac{2\sqrt{1-x^2}}{\pi} \quad -1<x<1$$

$$E(XY)=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{xy}{\pi}\,dx\,dy$$

$$=\int_{-1}^{1}\frac{y}{\pi}\left[\frac{x^2}{2}\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}dy$$

$$=\frac{1}{\pi}\int_{-1}^{1} y(1-y^2)\,dy$$

$$=\frac{1}{\pi}\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_{-1}^{1}=0$$

$$E(X)=\int_{-1}^{1}\frac{2x\sqrt{1-x^2}}{\pi}dx=\frac{-1}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\,d(1-x^2)$$

$$=\frac{-1}{\pi}\ \frac{2}{3}(1-x^2)^{\frac{3}{2}}\Big|_{-1}^{1}=0$$

Similarly, $E(Y)=0$

$$\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)=0$$

but $f(x,y)\neq f_X(x)\,f_Y(y)$ $\therefore$ X and Y are not independent

b1.

$f(x,y)=2 \quad 0<x\le y\le 1$

$f_X(x)=\int_x^1 2\,dy=2(1-x) \quad 0\le x\le 1$

$f_Y(y)=\int_0^y 2\,dy=2y \quad 0\le y\le 1$

(a)

$E(X)=\int_0^1 2x(1-x)\,dx=2\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=\frac{1}{3}$

$E(Y)=\int_0^1 2y^2\,dy=\left[\frac{2}{3}y^3\right]_0^1=\frac{2}{3}$

$E(X^2)=\int_0^1 2x^2(1-x)\,dx=2\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{1}{6}$

$E(Y^2)=\int_0^1 2y^3\,dy=\left[\frac{2}{4}y^4\right]_0^1=\frac{1}{2}$

$Var(X)=\frac{1}{6}-\left(\frac{1}{3}\right)^2=\frac{1}{18}$

$Var(Y)=\frac{1}{2}-\left(\frac{2}{3}\right)^2=\frac{1}{18}$

$E(XY)=\int_0^1\int_0^y 2xy\,dx\,dy=\int_0^1 y\left[x^2\right]_0^y dy=\int_0^1 y^3\,dy=\frac{1}{4}.$

$Cov(X,Y)=\frac{1}{4}-\frac{1}{3}\cdot\frac{2}{3}=\frac{1}{36}$

$Cor(X,Y)=\frac{\frac{1}{36}}{\sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}}=\frac{1}{2}$

sign of $Cor(X,Y)>0$. 這是因為 $Y>X$ 故 X 越大時 Y 也會傾向值越大 $\Rightarrow$ positive correlated

(b) $f_{X|Y}(x|y)=\frac{2}{2y}=\frac{1}{y} \quad 0\le x\le y\le 1 \Rightarrow X|Y=y\sim U(0,y)$

$f_{Y|X}(y|x)=\frac{2}{2(1-x)}=\frac{1}{1-x} \quad 0\le x\le y\le 1 \Rightarrow Y|X=x\sim U(x,1)$

$E(X|Y=y)=\int_0^y \frac{x}{y}\,dx=\frac{1}{y}\left[\frac{x^2}{2}\right]_0^y=\frac{y}{2} \quad 0\le y\le 1$

$E(Y|X=x)=\int_x^1 \frac{y}{1-x}\,dy=\frac{1}{1-x}\left.\frac{y^2}{2}\right|_x^1=\frac{1-x^2}{2(1-x)}=\frac{1+x}{2} \quad 0\le x\le 1$

make sense intuitively $\because X|Y=y\sim U(0,y) \quad Y|X=x\sim U(x,1)$

(c)

$Z = E(X|Y) = \frac{Y}{2}$ $W = E(Y|X) = \frac{1+X}{2}$

For $0 < z < \frac{1}{2}$ $P(Z \le z) = P(Y \le 2z) = \int_0^{2z} 2y\,dy = y^2 \Big|_0^{2z} = 4z^2$

$$f_Z(z) = \begin{cases} 8z & 0 < z < \frac{1}{2} \\ 0 & \text{o.w.} \end{cases}$$

For $\frac{1}{2} < w < 1$ $P(W \le w) = P\left(\frac{1+X}{2} < w\right) = P(X < 2w-1)$

$$= \int_0^{2w-1} 2(1-x)\,dx$$

$$= \left[2x - x^2\right]_0^{2w-1}$$

$$= 2(2w-1) - (2w-1)^2$$

$$= -4w^2 + 4w - 1 + 4w - 2$$

$$= -4w^2 + 8w - 3$$

$$f_W(w) = \begin{cases} 8(1-w) & \frac{1}{2} < w < 1 \\ 0 & \text{o.w.} \end{cases}$$

(d)

$a = \mu_Y - b\mu_X$ $b = \rho\frac{\sigma_Y}{\sigma_X}$

$= \frac{2}{3} - \frac{1}{3}b$ $= \frac{1}{2} \cdot 1 = \frac{1}{2}$

$= \frac{1}{2}$

$\Rightarrow Y = \frac{1}{2}(1+X)$

min MSE is $\sigma_Y^2(1-\rho^2) = \frac{1}{18}\left(1 - \frac{1}{2}\right) = \frac{1}{24}$

(c) by (b) $h(X) = E(Y|X) = \frac{1}{2} + \frac{1}{2}X$ [注意：在此例中，best predictor 和 (b) 中的 best linear predictor 是相同的]

$$\min MSE = E(\mathrm{Var}(Y|X)) = \mathrm{Var}(Y) - \mathrm{Var}[E(Y|X)]$$
$$= \frac{1}{18} - \mathrm{Var}\left(\frac{1+X}{2}\right)$$
$$= \frac{1}{18} - \frac{1}{4}\cdot\frac{1}{18}$$
$$= \frac{1}{24}.$$

12.

Z_1 = number in 1st generation

Z_2 = " 2nd "

Z_3 = " 3rd "

$E(Z_1) = \mu \quad \mathrm{Var}(Z_1) = \sigma^2$

$Z_2 = \sum_{i=1}^{Z_1} X_i \qquad X_i \overset{iid}{\sim} (\mu, \sigma^2)$

$E(Z_2|Z_1) = Z_1\mu$ and $\mathrm{Var}(Z_2|Z_1) = Z_1\sigma^2$

$E(Z_2) = E[E(Z_2|Z_1)] \quad E(Z_1\mu) = \mu^2$

$\mathrm{Var}(Z_2) = E[\mathrm{Var}(Z_2|Z_1)] + \mathrm{Var}[E(Z_2|Z_1)]$

$= E(Z_1\sigma^2) + \mathrm{Var}(Z_1\mu)$

$= \sigma^2\mu + \mu^2\sigma^2 = (\mu+\mu^2)\sigma^2$

$$Z_3 = \sum_{i=1}^{Z_2} X_i$$

$E(Z_3|Z_2) = Z_2\mu$ and $\mathrm{Var}(Z_3|Z_2) = Z_2\sigma^2$

$$E(Z_3) = E[E(Z_3|Z_2)] = E(Z_2\mu) = \mu^3$$

$$\mathrm{Var}(Z_3) = E[\mathrm{Var}(Z_3|Z_2)] + \mathrm{Var}[E(Z_3|Z_2)]$$

$$= E(\sigma^2 Z_2) + \mathrm{Var}(Z_2\mu)$$

$$= \sigma^2\mu^2 + \mu^2(\mu+\mu^2)\sigma^2$$

$$= \sigma^2(\mu^2+\mu^3+\mu^4)$$