Solution Manual **Homework 11**

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Problem 12 (Chapter 9)

Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with the density function $f(x|\theta) = \theta \exp[-\theta x]$. Derive a likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$, and show that the rejection region is of the form $\{\bar{X} \exp[-\theta_0 \bar{X}] \leq c\}$.

Solution:

First, we construct the likelihood function for the sample:

$$L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n \exp\left(-\theta \sum_{i=1}^{n} x_i\right) = \theta^n \exp(-\theta n \bar{X})$$

The generalized likelihood ratio is defined as:

$$\Lambda = \frac{\max_{\theta \in \omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)}$$

Numerator (Under H_0):

Under H_0, θ can only be θ_0. Thus, the maximum likelihood under H_0 is simply:

$$L(\theta_0) = \theta_0^n \exp(-\theta_0 n \bar{X})$$

Denominator (Unrestricted):

To find the maximum likelihood over the entire parameter space Ω, we find the Maximum Likelihood Estimator (MLE) of θ. The log-likelihood is:

$$\ell(\theta) = \ln L(\theta) = n \ln \theta - \theta n \bar{X}$$

Setting the derivative to zero:

$$\frac{d\ell(\theta)}{d\theta} = \frac{n}{\theta} - n\bar{X} = 0 \implies \hat{\theta} = \frac{1}{\bar{X}}$$

(We can verify it's a maximum since $d^2\ell/d\theta^2 = -n/\theta^2 < 0$). Substituting $\hat{\theta}$ back into the likelihood gives the denominator:

$$L(\hat{\theta}) = \left(\frac{1}{\bar{X}}\right)^n \exp\left(-\frac{1}{\bar{X}} n \bar{X}\right) = \left(\frac{1}{\bar{X}}\right)^n e^{-n}$$

Likelihood Ratio Test Statistic:

Now, we compute the likelihood ratio Λ:

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\theta_0^n \exp(-\theta_0 n \bar{X})}{(1/\bar{X})^n e^{-n}} = (\theta_0 \bar{X})^n \exp(n - \theta_0 n \bar{X}) = e^n \left[\theta_0 \bar{X} \exp(-\theta_0 \bar{X})\right]^n$$

The likelihood ratio test rejects H_0 for small values of Λ, i.e., $\Lambda \leq k$ for some constant $k \in (0, 1)$:

$$e^n \left[\theta_0 \bar{X} \exp(-\theta_0 \bar{X})\right]^n \leq k$$

$$\left[\theta_0 \bar{X} \exp(-\theta_0 \bar{X})\right]^n \leq k e^{-n}$$

Since taking the n-th root preserves the inequality for positive quantities:

$$\theta_0 \bar{X} \exp(-\theta_0 \bar{X}) \leq (k e^{-n})^{1/n}$$

Divide both sides by θ_0 (which is a positive constant):

$$\bar{X} \exp(-\theta_0 \bar{X}) \leq \frac{(k e^{-n})^{1/n}}{\theta_0}$$

Let $c = \frac{(k e^{-n})^{1/n}}{\theta_0}$. We have shown that the rejection region is of the form:

$$\left\{\bar{X} \exp[-\theta_0 \bar{X}] \leq c\right\}$$

Problem 13 (Chapter 9)
Suppose, to be specific, that in Problem 12, $\theta_0 = 1$, $n = 10$, and that $\alpha = .05$. In order to use the test, we must find the appropriate value of c.
a. Show that the rejection region is of the form $\{\bar{X} \le x_0\} \cup \{\bar{X} \ge x_1\}$, where x_0 and x_1 are determined by c.
b. Explain why c should be chosen so that $P(\bar{X}\exp(-\bar{X}) \le c) = .05$ when $\theta_0 = 1$.
c. Explain why $\sum_{i=1}^{10} X_i$ and hence $\bar{X}$ follow gamma distributions when $\theta_0 = 1$. How could this knowledge be used to choose c?

Solution:

Part (a):

From Problem 12, the rejection region is $\bar{X}\exp(-\theta_0\bar{X}) \le c$. Given $\theta_0 = 1$, this becomes $\bar{X}\exp(-\bar{X}) \le c$. Let $g(x) = xe^{-x}$ for $x > 0$. Let's analyze the shape of this function by taking its derivative:

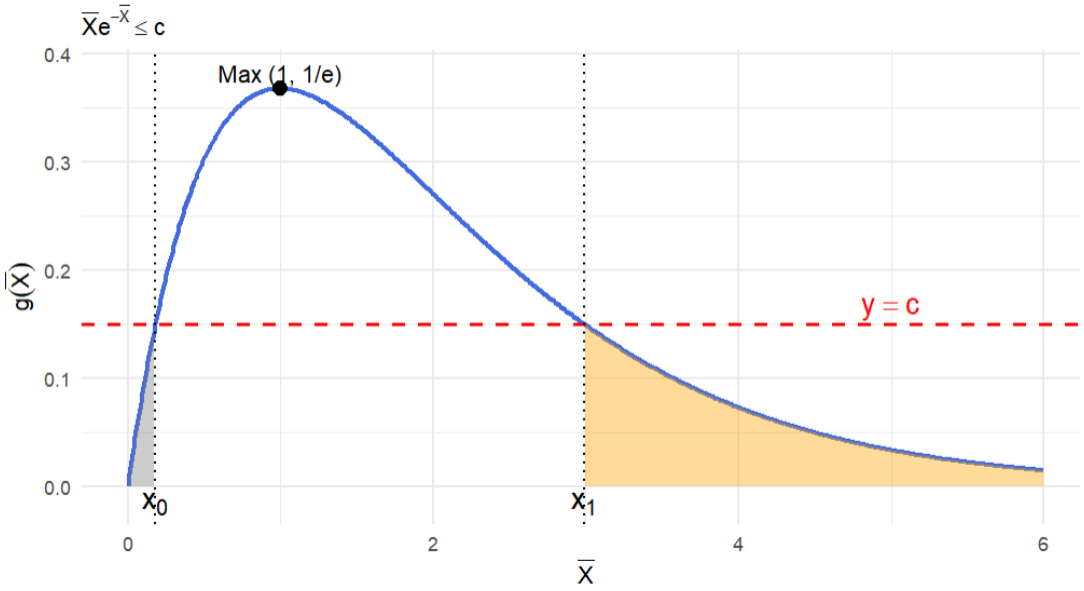
$$g'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$$

Setting $g'(x) = 0$, we find a critical point at $x = 1$. For $x < 1$, $g'(x) > 0$ (function is increasing). For $x > 1$, $g'(x) < 0$ (function is decreasing). Thus, $g(x)$ is a unimodal function that reaches its global maximum at $x = 1$ (where $g(1) = 1/e$).

Because $g(x)$ increases on $(0, 1)$ and decreases on $(1, \infty)$, drawing a horizontal line $y = c$ (where $0 < c < 1/e$) will intersect the curve $y = g(x)$ at exactly two points, say x_0 and x_1, with $x_0 < 1 < x_1$. Therefore, the inequality $xe^{-x} \le c$ is satisfied when $x \le x_0$ or when $x \ge x_1$. Replacing x with $\bar{X}$, the rejection region is of the form:

$$\{\bar{X} \le x_0\} \cup \{\bar{X} \ge x_1\}$$

where x_0 and x_1 are the solutions to the equation $xe^{-x} = c$.



Part (b):

The significance level α of a hypothesis test is defined as the probability of rejecting the null hypothesis H_0 when H_0 is actually true (Type I error rate).

$$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true})$$

We are given $\alpha = 0.05$. From Problem 12, we know the rule to reject H_0 is when $\bar{X}\exp(-\theta_0\bar{X}) \le c$. Under H_0, we have $\theta_0 = 1$. Therefore, to control the Type I error rate at 0.05, we must choose the critical value c such that:

$$P(\bar{X}\exp(-\bar{X}) \le c \mid \theta = 1) = 0.05$$

Part (c):

When $\theta_0 = 1$, each observation X_i follows an exponential distribution with rate parameter 1, i.e., $X_i \sim \text{Exp}(1)$. An Exponential distribution with rate 1 is equivalent to a Gamma distribution with shape parameter $\alpha = 1$ and rate parameter $\lambda = 1$, i.e., $X_i \sim \text{Gamma}(1, 1)$. Since $X_1, \ldots, X_{10}$ are independent and identically distributed, their sum $S = \sum_{i=1}^{10} X_i$ has a Gamma distribution where the shape parameters add up:

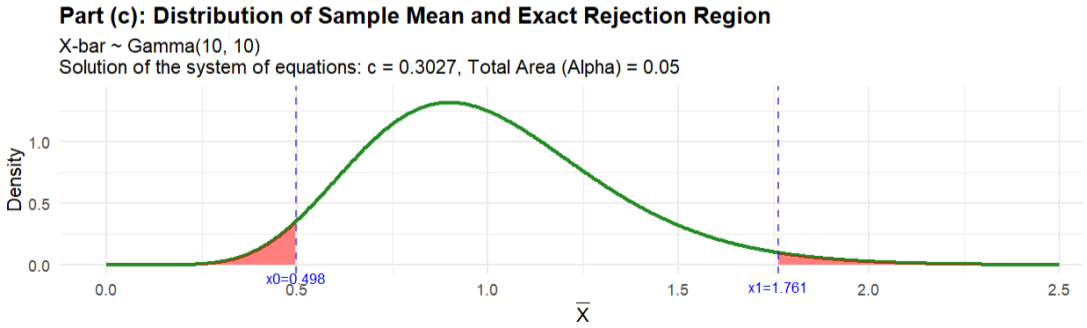
$$S = \sum_{i=1}^{10} X_i \sim \text{Gamma}(10, 1)$$

The sample mean is $\bar{X} = S/10$. By the scaling property of the Gamma distribution, multiplying a $\text{Gamma}(\alpha, \lambda)$ random variable by a constant $k = 1/10$ results in a $\text{Gamma}(\alpha, \lambda/k)$ distribution. Thus:

$$\bar{X} \sim \text{Gamma}(10, 10)$$

How to use this to choose c: From part (a), we know rejecting H_0 is equivalent to $\bar{X} \leq x_0$ or $\bar{X} \geq x_1$. To satisfy $\alpha = 0.05$, we need $P(\bar{X} \leq x_0) + P(\bar{X} \geq x_1) = 0.05$. Additionally, from the definition of the rejection region curve $xe^{-x} = c$, x_0 and x_1 must satisfy:

$$x_0 e^{-x_0} = x_1 e^{-x_1} = c$$



Using the known Probability Density Function (or CDF) of the Gamma(10, 10) distribution, we can set up a system of two equations: 1. $\int_0^{x_0} f_{\bar{X}}(x)dx + \int_{x_1}^{\infty} f_{\bar{X}}(x)dx = 0.05$ 2. $x_0 e^{-x_0} = x_1 e^{-x_1}$
Solving this system numerically gives us x_0 and x_1, and substituting either back into $c = x_0 e^{-x_0}$ will determine the exact value of c. (Alternatively, one could simplify by using equal tail probabilities, setting each tail to 0.025, though this only approximates the exact likelihood ratio test).

Problem 24 (Chapter 9)
Let X be a binomial random variable with n trials and probability p of success.
a. What is the generalized likelihood ratio for testing $H_0 : p = .5$ versus $H_A : p \neq .5$?
b. Show that the test rejects for large values of $|X - n/2|$.
c. Using the null distribution of X, show how the significance level corresponding to a rejection region $|X - n/2| > k$ can be determined.
d. If $n = 10$ and $k = 2$, what is the significance level of the test?
e. Use the normal approximation to the binomial distribution to find the significance level if $n = 100$ and $k = 10$.

[**Hint 1:** For (b), you can first show that (i) the likelihood ratio $\Lambda(X)$ is symmetric about $n/2$, i.e., $\Lambda(n/2 - t) = \Lambda(n/2 + t)$, for $0 < t < n/2$, and (ii) $g(t) \equiv \log \Lambda(n/2 + t)$ is a decreasing function as t increases, where $0 < t < n/2$. **Hint 2:** For (d), if $Y \sim B(10, 0.5)$, then $P(Y \in \{0, 1, 2\}) = 0.055$. **Hint 3:** For (e), apply central limit theorem, see LN, Ch1-6, p.97, example 5.6.]

Solution:

Part (a):

The likelihood function for a Binomial distribution is:

$$L(p) = \binom{n}{X} p^X (1-p)^{n-X}$$

Under H_0, $p = 0.5$, so the numerator of the GLR test statistic is:

$$L(0.5) = \binom{n}{X} (0.5)^X (1-0.5)^{n-X} = \binom{n}{X} \left(\frac{1}{2}\right)^n$$

For the denominator of the GLR test statistic, the unconstrained MLE for p is $\hat{p} = \frac{X}{n}$. Thus:

$$L(\hat{p}) = \binom{n}{X} \left(\frac{X}{n}\right)^X \left(1 - \frac{X}{n}\right)^{n-X} = \binom{n}{X} \left(\frac{X}{n}\right)^X \left(\frac{n-X}{n}\right)^{n-X}$$

The generalized likelihood ratio is:

$$\Lambda(X) = \frac{L(0.5)}{L(\hat{p})} = \frac{\binom{n}{X}\left(\frac{1}{2}\right)^n}{\binom{n}{X}\left(\frac{X}{n}\right)^X \left(\frac{n-X}{n}\right)^{n-X}} = \frac{n^n/2^n}{X^X (n-X)^{n-X}} = \frac{(n/2)^n}{X^X (n-X)^{n-X}}$$

Part (b):

We reject H_0 if $\Lambda(X) \leq c$. Following **Hint 1**:
(i) Let's check symmetry around $n/2$. Substitute $X = n/2 + t$:

$$\Lambda(n/2 + t) = \frac{(n/2)^n}{(n/2+t)^{n/2+t}(n - (n/2+t))^{n-(n/2+t)}} = \frac{(n/2)^n}{(n/2+t)^{n/2+t}(n/2-t)^{n/2-t}}$$

Now substitute $X = n/2 - t$:

$$\Lambda(n/2 - t) = \frac{(n/2)^n}{(n/2-t)^{n/2-t}(n - (n/2-t))^{n-(n/2-t)}} = \frac{(n/2)^n}{(n/2-t)^{n/2-t}(n/2+t)^{n/2+t}}$$

Clearly, $\Lambda(n/2 + t) = \Lambda(n/2 - t)$, so $\Lambda(X)$ is symmetric about $n/2$.
(ii) Let $g(t) = \log \Lambda(n/2 + t)$ for $0 < t < n/2$.

$$g(t) = \log\left(\left(\frac{n}{2}\right)^n\right) - \left(\frac{n}{2} + t\right)\log\left(\frac{n}{2} + t\right) - \left(\frac{n}{2} - t\right)\log\left(\frac{n}{2} - t\right)$$

Take the derivative with respect to t:

$$\begin{aligned} g'(t) &= 0 - \left[1 \cdot \log\left(\frac{n}{2}+t\right) + \left(\frac{n}{2}+t\right)\frac{1}{\frac{n}{2}+t}\right] - \left[-1 \cdot \log\left(\frac{n}{2}-t\right) + \left(\frac{n}{2}-t\right)\frac{-1}{\frac{n}{2}-t}\right] \\ &= -\left[\log\left(\frac{n}{2}+t\right)+1\right] + \left[\log\left(\frac{n}{2}-t\right)+1\right] \\ &= -\log\left(\frac{n}{2}+t\right) + \log\left(\frac{n}{2}-t\right) = \log\left(\frac{n/2-t}{n/2+t}\right) \end{aligned}$$

Since $t > 0$, we have $n/2 - t < n/2 + t$. Therefore, the fraction $\frac{n/2-t}{n/2+t} < 1$, which means its logarithm is negative. Thus, $g'(t) < 0$, meaning $g(t)$ is a strictly decreasing function of t for $t \in (0, n/2)$. Since logarithmic transformation is monotonic, $\Lambda(n/2+t)$ is also strictly decreasing as t increases.
Conclusion: Because $\Lambda(X)$ is symmetric about $n/2$ and decreases as the distance from the center $t = |X - n/2|$ increases, the condition $\Lambda(X) \le c$ is equivalent to the condition that the distance t is large. Therefore, the test rejects for large values of $|X - n/2|$, or $|X - n/2| > k$ for some constant k.

Part (c):
The significance level α is the probability of rejecting H_0 when H_0 is true. Under H_0, $X \sim B(n, 0.5)$. The rejection region is $|X - n/2| > k$. Therefore, α can be determined by summing the probabilities of the binomial distribution in the rejection tails:

$$\alpha = P\left(|X - n/2| > k \mid p = 0.5\right) = \sum_{x:|x-n/2|>k} \binom{n}{x} (0.5)^n$$

Part (d):
Given $n = 10$, $k = 2$. Under H_0, $X \sim B(10, 0.5)$. The rejection region is:

$$|X - 5| > 2 \implies X - 5 > 2 \quad \text{or} \quad X - 5 < -2 \implies X > 7 \quad \text{or} \quad X < 3$$

Since X is an integer, this corresponds to $X \in \{0, 1, 2, 8, 9, 10\}$. Using **Hint 2**, if $Y \sim B(10, 0.5)$, then $P(Y \le 2) = P(Y \in \{0, 1, 2\}) = 0.055$. By the symmetry of the Binomial distribution when $p = 0.5$, $P(X \ge 8) = P(X \in \{8, 9, 10\}) = P(X \le 2) = 0.055$. Thus, the significance level is:

$$\alpha = P(X \le 2) + P(X \ge 8) = 0.055 + 0.055 = 0.11$$

Part (e):
Given $n = 100$, $k = 10$. Under H_0, $X \sim B(100, 0.5)$. The rejection region is:

$$|X - 50| > 10 \implies X > 60 \quad \text{or} \quad X < 40 \implies X \ge 61 \quad \text{or} \quad X \le 39$$

Using **Hint 3** (Central Limit Theorem), for large n, X can be approximated by a Normal distribution:

$$X \approx N(np, np(1-p)) = N(100(0.5), 100(0.5)(0.5)) = N(50, 25)$$

The standard deviation is $\sigma = \sqrt{25} = 5$. Using continuity correction for better accuracy:

$$\begin{aligned} \alpha &\approx P(X \le 39.5) + P(X \ge 60.5) \\ &= P\left(\frac{X-50}{5} \le \frac{39.5-50}{5}\right) + P\left(\frac{X-50}{5} \ge \frac{60.5-50}{5}\right) \\ &= P(Z \le -2.1) + P(Z \ge 2.1) \end{aligned}$$

where $Z \sim N(0, 1)$.

From the standard normal table, $P(Z \le -2.1) \approx 0.0179$. By symmetry, $P(Z \ge 2.1) \approx 0.0179$.

$$\alpha \approx 0.0179 + 0.0179 = 0.0358$$

(Note: If continuity correction is not applied, we calculate $P(|Z| > \frac{10}{5}) = P(|Z| > 2) = 2 \times 0.0228 = 0.0456$.)

Problem 26 (Chapter 9)
True or false. Also if false, please explain why.
a. The generalized likelihood ratio statistic Λ is always less than or equal to 1.
b. If the p-value is .03, the corresponding test will reject at the significance level .02.
c. If a test rejects at significance level .06, then the p-value is less than or equal to .06.
d. The p-value of a test is the probability that the null hypothesis is correct.
e. In testing a simple versus simple hypothesis via the likelihood ratio, the p-value equals the likelihood ratio.
f. If a chi-square test statistic with 4 degrees of freedom has a value of 8.5, the p-value is less than .05.

Solution:

Part (a): True.

The generalized likelihood ratio is $\Lambda = \frac{\max_{\theta\in\omega_0} L(\theta)}{\max_{\theta\in\Omega} L(\theta)}$. Since the null parameter space ω_0 is a subset of the entire parameter space Ω (i.e., $\omega_0 \subseteq \Omega$), the maximum of the likelihood function over the restricted set ω_0 can never exceed its maximum over the larger set Ω. Hence, the numerator is always $\leq$ the denominator, making $\Lambda \leq 1$.

Part (b): False.

Explanation: The rule for rejection is to reject H_0 if the p-value is less than or equal to the significance level α. In this case, the p-value is 0.03 and α is 0.02. Since $0.03 > 0.02$, we **fail to reject** H_0.

Part (c): True.

A test rejects H_0 at a given significance level α if and only if the p-value $\leq \alpha$. If it rejects at $\alpha = 0.06$, it mathematically guarantees that the p-value is ≤ 0.06.

Part (d): False.

Explanation: The p-value is defined as the probability of obtaining test results at least as extreme as the results actually observed, *under the assumption that the null hypothesis is true.* Under the Frequentist approach, the probability that the null hypothesis is correct is either 1 (H_0 is true) or 0 (H_0 is false & H_A is true).

Part (e): False.

Explanation: The p-value and the likelihood ratio are two entirely different mathematical entities. The likelihood ratio (Λ) is a ratio of likelihood functions (which can be probability mass or density evaluated at specific data points). The p-value is an area (an integral or sum) under the distribution curve of the test statistic under H_0, representing the probability of seeing a more extreme test statistic. They are not equal.

Part (f): False.

Explanation: We need to check the critical value of a χ^2 distribution with 4 degrees of freedom at $\alpha = 0.05$. Looking at a Chi-square table, the 95th percentile (area to the right is 0.05) for 4 degrees of freedom is $\chi^2_{4,0.05} \approx 9.488$. Because our test statistic value is 8.5, which is less than 9.488 (8.5 is not in the extreme 5% tail), the area to the right of 8.5 is strictly greater than 0.05. Therefore, the p-value is **greater than** 0.05.

Problem 37

We are given the number of deaths due to accidental falls for each month during 1970. We want to test if there is any evidence for a departure from uniformity in the death rate over time (i.e., whether a seasonal pattern exists).

Hypothesis Testing:
Let X_i be the number of deaths in month i, for $i = 1, 2, \ldots, 12$. Based on the hint, we model X_i as independent random variables, each distributed as Poisson with parameter λ_i. We want to examine whether all λ_i's are equal.

- Null Hypothesis (H_0): $\lambda_1 = \lambda_2 = \cdots = \lambda_{12} = \lambda$ (The death rate is uniform over time).
- Alternative Hypothesis (H_A): Not all λ_i are equal (There is a seasonal pattern).

Test Statistic (Pearson's Chi-Square Test):
Under the null hypothesis H_0, the maximum likelihood estimate (MLE) of the common rate λ is the sample mean of the monthly deaths. The total number of deaths in 1970 is:

$$N = \sum_{i=1}^{12} X_i = 1668 + 1407 + \cdots + 1526 = 16916$$

Thus, the expected number of deaths for each month under H_0 is:

$$\hat{\lambda} = \bar{X} = \frac{16916}{12} \approx 1409.67$$

By the Example 7.22 in LNp.4-49, we can use the Pearson's Chi-square test statistic for goodness-of-fit:

$$\chi^2 = \sum_{i=1}^{12} \frac{(X_i - \hat{\lambda})^2}{\hat{\lambda}}$$

Let's calculate the components:

$$\begin{aligned}
\chi^2 &= \frac{(1668 - 1409.67)^2}{1409.67} + \frac{(1407 - 1409.67)^2}{1409.67} + \cdots + \frac{(1526 - 1409.67)^2}{1409.67} \\
&= \frac{66736.11}{1409.67} + \frac{7.11}{1409.67} + \cdots + \frac{13533.44}{1409.67} \\
&\approx 47.34 + 0.01 + 1.12 + 7.19 + 3.34 + 3.64 + 0.01 + 0.94 + 4.28 + 1.55 + 0.00 + 9.60 \\
&\approx 79.01
\end{aligned}$$

Conclusion:
Under H_0, the test statistic χ^2 follows a Chi-square distribution with degrees of freedom $df = k - 1 = 12 - 1 = 11$.
For a significance level of $\alpha = 0.05$, the critical value is $\chi^2_{11,0.05} \approx 19.675$.

Since our calculated test statistic $\chi^2 \approx 79.01$ is much greater than 19.675 (and the p-value is extremely small), we **reject the null hypothesis** H_0. There is strong statistical evidence of a departure from uniformity in the rate over time.

Problem 40

Consider testing goodness of fit for a multinomial distribution with two cells. The number of observations are X_1 and X_2, and the hypothesized probabilities are p_1 and p_2. We want to show that Pearson's chi-square statistic:

$$\chi^2 = \sum_{i=1}^{2} \frac{(X_i - np_i)^2}{np_i}$$

can be expressed as:

$$\frac{(X_1 - np_1)^2}{np_1(1-p_1)}$$

Proof:

First, we write out the summation for the two cells ($i = 1$ and $i = 2$):

$$\chi^2 = \frac{(X_1 - np_1)^2}{np_1} + \frac{(X_2 - np_2)^2}{np_2}$$

From the properties of a two-cell multinomial (binomial) distribution, we know the total number of observations and the sum of probabilities must satisfy:

1. $X_1 + X_2 = n \implies X_2 = n - X_1$
2. $p_1 + p_2 = 1 \implies p_2 = 1 - p_1$

Let's substitute X_2 and p_2 in the numerator of the second term:

$$X_2 - np_2 = (n - X_1) - n(1 - p_1)$$

$$X_2 - np_2 = n - X_1 - n + np_1$$

$$X_2 - np_2 = -(X_1 - np_1)$$

Squaring this result gives us:

$$(X_2 - np_2)^2 = [-(X_1 - np_1)]^2 = (X_1 - np_1)^2$$

Now, substitute this squared term back into our expanded χ^2 equation:

$$\chi^2 = \frac{(X_1 - np_1)^2}{np_1} + \frac{(X_1 - np_1)^2}{np_2}$$

We can factor out the common numerator $(X_1 - np_1)^2$:

$$\chi^2 = (X_1 - np_1)^2 \left(\frac{1}{np_1} + \frac{1}{np_2}\right)$$

Next, let's simplify the expression inside the parentheses:

$$\frac{1}{np_1} + \frac{1}{np_2} = \frac{1}{n}\left(\frac{1}{p_1} + \frac{1}{p_2}\right)$$

$$= \frac{1}{n}\left(\frac{p_2 + p_1}{p_1 p_2}\right)$$

Since $p_1 + p_2 = 1$, the numerator becomes 1:

$$= \frac{1}{n}\left(\frac{1}{p_1 p_2}\right) = \frac{1}{np_1 p_2}$$

Substitute $p_2 = 1 - p_1$ into the denominator:

$$= \frac{1}{np_1(1-p_1)}$$

Finally, multiply this simplified term back with our factored numerator:

$$\chi^2 = \frac{(X_1 - np_1)^2}{np_1(1-p_1)}$$

This completes the proof. Since X_1 is binomially distributed, $\frac{X_1 - np_1}{\sqrt{np_1(1-p_1)}}$ approximately follows a standard normal distribution $N(0,1)$ under the null hypothesis. Therefore, its square (our derived statistic) is approximately distributed as a chi-square random variable with 1 degree of freedom.

Problem 43

(a)

A student reported getting 9207 heads and 8743 tails in 17,950 coin tosses. Is this a significant discrepancy from the null hypothesis $H_0 : p = \frac{1}{2}$?

Solution 1:
We can use GLR test in LNp.4-36∼38 or Pearson's Chi-square test in LNp.4-41∼42 to evaluate the discrepancy. Statisical modeling $X_1, \cdots, X_{17950} \overset{\text{iid}}{\sim} \text{Bernoulli}(p_1) \Rightarrow (O_{Heads}, O_{Tails}) \sim \text{Multinomial}(17950, p_1, p_2)$. The test involves two cells: Heads and Tails.

- $H_0 : p_1 = 0.5$ (The coin is fair)
- $H_A : p_1 \neq 0.5$ (The coin is not fair)

The total number of tosses is $n = 17950$. Under H_0, the expected frequencies for both heads and tails are:

$$E_{\text{Heads}} = E_{\text{Tails}} = 17950 \times 0.5 = 8975$$

The observed frequencies are $O_{\text{Heads}} = 9207$ and $O_{\text{Tails}} = 8743$. The Pearson's Chi-square test statistic is calculated as follows:

$$\begin{aligned}\chi^2 &= \frac{(9207 - 8975)^2}{8975} + \frac{(8743 - 8975)^2}{8975} \\ &= \frac{232^2}{8975} + \frac{(-232)^2}{8975} \\ &= \frac{53824}{8975} + \frac{53824}{8975} \approx 5.997 + 5.997 = 11.994\end{aligned}$$

Under the null hypothesis H_0, the test statistic follows a Chi-square distribution with degrees of freedom $df = k - 1 = 1 - 0 = 1$. ← because $\Omega = \{(p_1, p_2) \mid 0 < p_1, p_2 < 1, p_1 + p_2 = 1\} \Rightarrow \dim(\Omega) = 1$; $\Omega_0 = \{(p_1, p_2) \mid p_1 = p_2 = 1/2\} \Rightarrow \dim(\Omega_0) = 0$

For a significance level of $\alpha = 0.05$, the critical value is $\chi^2_{1,0.05} = 3.841$.

Since our calculated test statistic $\chi^2 \approx 11.994$ is greater than the critical value of 3.841 (with a p-value < 0.001), we **reject the null hypothesis** H_0. There is strong statistical evidence indicating a significant discrepancy from the assumption of a fair coin.

Solution 2: Generalized Likelihood Ratio Test (GLRT)

Alternatively, we can approach this using the Generalized Likelihood Ratio Test in LNp.4-36∼38.

By Wilks' Theorem, the GLRT statistic

$$-2\log\Lambda = 2\sum_{i=1}^{2} O_i \log\left(\frac{O_i}{E_i}\right)$$

asymptotically follows a Chi-square distribution.

Substituting our observed counts ($O_1 = 9207, O_2 = 8743$) and expected counts ($E_1 = E_2 = 8975$):

$$\begin{aligned} -2\log\Lambda &= 2\left[9207\log\left(\frac{9207}{8975}\right) + 8743\log\left(\frac{8743}{8975}\right)\right] \\ &\approx 2\left[9207(0.02552) + 8743(-0.02617)\right] \\ &\approx 2\left[234.96 - 228.80\right] \\ &\approx 2 \times 6.16 = 12.32 \end{aligned}$$

Conclusion for GLRT:

Under H_0, the statistic $-2\log\Lambda \sim \chi^2(df = 1)$. For a significance level of $\alpha = 0.05$, the critical value is $\chi^2_{1,0.05} = 3.841$.

Since our calculated GLR test statistic $-2\log\Lambda \approx 12.32$ is significantly larger than the critical value of 3.841 (with p-value < 0.0005), we **reject the null hypothesis** H_0. Both Pearson's Chi-square test and the GLR test provide consistent and robust evidence that the observed coin tosses diverge significantly from the behavior of a fair coin.

(b)

The student tossed groups of five coins at a time. Total number of trials (groups) is $m = 100 + 524 + 1080 + 1126 + 655 + 105 = 3590$. Are the data consistent with the hypothesis that all the coins were fair ($p = \frac{1}{2}$)?

Solution:

Under the hypothesis that all coins are fair, the number of heads in a group of 5 follows a binomial distribution $B(5, 0.5)$: statisical modeling : $X_1, \cdots, X_{3590} \overset{\text{iid}}{\sim} \text{Binomial}(5, p) \Rightarrow (O_0, \cdots, O_5) \sim \text{Multinomial}(3590, p_0, ..., p_5)$ Under null, the probability of obtaining k heads is given by $P_{0,k} = P(X = k) = \binom{5}{k}(0.5)^5$. The expected frequency for each cell is computed as $E_k = m \times P(X = k) = 3590 \times \binom{5}{k}(0.5)^5$.

We construct the following table to summarize the observed and expected frequencies:

Number of Heads (k)	Observed (O_k)	Probability $P_{0,k} = P(X = k), under\ H_0$	Expected (E_k)
0	100	$1/32 = 0.03125$	$3590 \times 0.03125 = 112.1875$
1	524	$5/32 = 0.15625$	$3590 \times 0.15625 = 560.9375$
2	1080	$10/32 = 0.3125$	$3590 \times 0.3125 = 1121.875$
3	1126	$10/32 = 0.3125$	$3590 \times 0.3125 = 1121.875$
4	655	$5/32 = 0.15625$	$3590 \times 0.15625 = 560.9375$
5	105	$1/32 = 0.03125$	$3590 \times 0.03125 = 112.1875$
Total	**3590**	**1.0000**	**3590**

Now, we compute Pearson's Chi-square statistic:

$$\begin{aligned}
\chi^2 &= \sum_{k=0}^{5} \frac{(O_k - E_k)^2}{E_k} \\
&= \frac{(100 - 112.19)^2}{112.19} + \frac{(524 - 560.94)^2}{560.94} + \frac{(1080 - 1121.88)^2}{1121.88} \\
&\quad + \frac{(1126 - 1121.88)^2}{1121.88} + \frac{(655 - 560.94)^2}{560.94} + \frac{(105 - 112.19)^2}{112.19} \\
&\approx 1.324 + 2.432 + 1.564 + 0.015 + 15.772 + 0.461 \\
&\approx 21.568
\end{aligned}$$

Under H_0, since the parameters of the distribution are completely specified ($p = 0.5$), the degrees of freedom is df = number of cells $- 1 = 5 - 0 = 5$. $\leftarrow$ because$\Omega = \{(p_0, \cdots, p_5) \mid 0 < p_0, \cdots, p_5 < 1, p_0 + \cdots p_5 = 1\} \Rightarrow \dim(\Omega) = 5$; $\Omega_0 = \{(p_0, \cdots, p_5) \mid p_k = \binom{5}{k}(0.5)^5\} \Rightarrow \dim(\Omega_0) = 0$.

For $\alpha = 0.05$, the critical value is $\chi^2_{5,0.05} = 11.070$.

Since $\chi^2 \approx 21.568 > 11.070$, we **reject the null hypothesis**. This indicates that the observed data are not consistent with the hypothesis that the groups follow a binomial distribution with $p = 0.5$.

Solution 2: Generalized Likelihood Ratio Test (GLRT)

Alternatively, we can evaluate the goodness-of-fit using the Generalized Likelihood Ratio Test. Under the unrestricted multinomial framework Ω, the MLEs for the cell probabilities are given by the observed proportions $\hat{p}_k = O_k/m$. Under the restricted null hypothesis Ω_0, the probabilities are fully fixed by the fair binomial distribution $B(5, 0.5)$, where $p_{0k} = \binom{5}{k}(0.5)^5$.

The likelihood ratio test statistic $-2\log\Lambda$ is expressed as:

$$-2\log\Lambda = 2\sum_{k=0}^{5} O_k \log\left(\frac{O_k}{E_k}\right)$$

where the expected frequencies E_k are the precise values obtained from the null model (112.1875, 560.9375, and 1121.8750).

Substituting the observed counts O_k and the expected counts E_k into the formulation yields:

$$\begin{aligned}
-2\log\Lambda &= 2\left[100\log\left(\frac{100}{112.1875}\right) + 524\log\left(\frac{524}{560.9375}\right) + 1080\log\left(\frac{1080}{1121.8750}\right)\right. \\
&\quad \left. +1126\log\left(\frac{1126}{1121.8750}\right) + 655\log\left(\frac{655}{560.9375}\right) + 105\log\left(\frac{105}{112.1875}\right)\right] \\
&\approx 2\left[-11.5005 - 35.7022 - 41.0813 + 4.1322 + 101.5336 - 6.9451\right] \\
&= 2\left[10.4367\right] \\
&= 20.8734
\end{aligned}$$

Conclusion for GLRT:

For a significance level of $\alpha = 0.05$, the critical value from the Chi-square table is $\chi^2_{5,0.05} = 11.070$.

Since the calculated GLRT statistic $-2\log\Lambda \approx 20.8734$ is strictly greater than the critical value of 11.070 (with a precise p-value ≈ 0.0008), we **reject the null hypothesis** H_0. The generalized likelihood ratio test perfectly confirms the previous conclusion that the data significantly depart from the assumption of a fair coin model where $p = 0.5$.

(c)

Are the data consistent with the hypothesis that all five coins had the same probability of heads but that this probability was not necessarily $\frac{1}{2}$?

Solution:

對 (b) 中的 $(O_0, \ldots, O_5) \sim \text{multinomial}(3590, p_0, \ldots, p_5)$

將 null 跟 alternative 修改成：

$$H_0 : \Omega_0 \quad \text{vs} \quad H_A : \Omega \setminus \Omega_0$$

其中參數空間與其維度定義如下：

$$\Omega_0 = \left\{(p_0, \ldots, p_5) \mid p_k = \binom{5}{k} q^k (1-q)^{5-k},\ 0 < q < 1\right\} \implies \dim(\Omega_0) = 1$$

$$\Omega = \left\{(p_0, \ldots, p_5) \mid 0 < p_0, \ldots, p_5 < 1,\ \sum_{i=0}^{5} p_i = 1\right\} \implies \dim(\Omega) = 5$$

Under H_0, the success probability q is unknown but assumed to be identical across all coins. We first determine the Maximum Likelihood Estimate (MLE) of q under H_0. The total number of heads observed across all trials is:

$$\text{Total Heads} = (0 \times 100) + (1 \times 524) + (2 \times 1080) + (3 \times 1126) + (4 \times 655) + (5 \times 105) = 9207$$

The total number of individual coin tosses is $5 \times 3590 = 17950$. Thus, the MLE for q is:

$$\hat{q} = \frac{9207}{17950} \approx 0.512925$$

We recalculate the expected frequencies under the model $B(5, \hat{q})$ where $\hat{q} \approx 0.512925$:

$$\hat{P_{0,k}} = \hat{P}(X = k) = \binom{5}{k} \hat{q}^k (1 - \hat{q})^{5-k}$$

k	Observed (O_k)	New Probability $\hat{P_{0,k}} = \hat{P}(X = k), under\ H_0$	New Expected (E_k)
0	100	$\binom{5}{0}(0.5129)^0(0.4871)^5 = 0.02755$	$3590 \times 0.02755 = 98.90$
1	524	$\binom{5}{1}(0.5129)^1(0.4871)^4 = 0.14510$	$3590 \times 0.14510 = 520.91$
2	1080	$\binom{5}{2}(0.5129)^2(0.4871)^3 = 0.30568$	$3590 \times 0.30568 = 1097.39$
3	1126	$\binom{5}{3}(0.5129)^3(0.4871)^2 = 0.32199$	$3590 \times 0.32199 = 1155.94$
4	655	$\binom{5}{4}(0.5129)^4(0.4871)^1 = 0.16959$	$3590 \times 0.16959 = 608.83$
5	105	$\binom{5}{5}(0.5129)^5(0.4871)^0 = 0.03573$	$3590 \times 0.03573 = 128.27$
Total	**3590**	**1.0000**	**3590.26**

The updated Pearson's Chi-square statistic is calculated as:

$$\begin{aligned}\chi^2 &= \frac{(100 - 98.90)^2}{98.90} + \frac{(524 - 520.91)^2}{520.91} + \frac{(1080 - 1097.39)^2}{1097.39} \\ &\quad + \frac{(1126 - 1155.94)^2}{1155.94} + \frac{(655 - 608.83)^2}{608.83} + \frac{(105 - 128.27)^2}{128.27} \\ &\approx 0.012 + 0.018 + 0.275 + 0.775 + 3.501 + 4.221 \\ &\approx 8.802\end{aligned}$$

Under the null hypothesis, because we estimated one parameter ($\boldsymbol{q}$) from the data sample:

$$df = \dim(\Omega) - \dim(\Omega_0) = 5 - 1 = 4$$

For a significance level of $\alpha = 0.05$, the critical value is $\chi^2_{4,0.05} = 9.488$.

Since our test statistic $\chi^2 \approx 8.802$ is less than the critical value of 9.488, we **fail to reject the null hypothesis**. This demonstrates that the data are consistent with the hypothesis that all five coins share the same probability of landing heads, even though that probability is not exactly 0.5.

Solution 2: Generalized Likelihood Ratio Test (GLRT)

We can also evaluate the goodness-of-fit for the unspecified binomial model using the Generalized Likelihood Ratio Test. Under the restricted sub-space Ω_0, the success probability p is estimated via the Maximum Likelihood Estimator (MLE):

$$\hat{q} = \frac{9207}{17950} \approx 0.512925$$

The cell probabilities under H_0 are given by $\hat{P_{0,k}} = \binom{5}{k}\hat{q}^k(1 - \hat{q})^{5-k}$, yielding the updated expected frequencies $E_k = m \cdot \hat{P_{0,k}}$.

Under the unrestricted multinomial framework Ω, the cell probabilities are estimated by the sample

proportions $\hat{p}_k = O_k/m$. The log-likelihood ratio test statistic $-2\log\Lambda$ is formulated as:

$$-2\log\Lambda = 2\sum_{k=0}^{5} O_k \log\left(\frac{O_k}{E_k}\right)$$

Substituting the observed frequencies O_k and the estimated expected frequencies E_k into the equation:

$$\begin{aligned}
-2\log\Lambda &= 2\left[100\log\left(\frac{100}{98.8957}\right) + 524\log\left(\frac{524}{520.9168}\right) + 1080\log\left(\frac{1080}{1097.3912}\right)\right. \\
&\quad \left. +1126\log\left(\frac{1126}{1155.9189}\right) + 655\log\left(\frac{655}{608.8252}\right) + 105\log\left(\frac{105}{128.0522}\right)\right] \\
&\approx 2\left[1.1105 + 3.0922 - 17.2558 - 29.5301 + 47.8427 - 21.0003\right] \\
&= 2\left[4.2592\right] \\
&= 8.5195
\end{aligned}$$

Conclusion for GLRT:

For a significance level of $\alpha = 0.05$, the critical value from the Chi-square table is $\chi^2_{4,0.05} = 9.488$.

Since the calculated test statistic $-2\log\Lambda \approx 8.5195$ is strictly less than the critical value of 9.488 (with a corresponding p-value ≈ 0.0743), we **fail to reject the null hypothesis** H_0. The GLRT reinforces the Pearson's chi-square test conclusion, indicating that the experimental data remain fully consistent with the hypothesis that all five coins share an identical success probability p.

Problem 44

Derive and carry out a likelihood ratio test (LRT) of the hypothesis $H_0 : \theta = \frac{1}{2}$ versus $H_1 : \theta \neq \frac{1}{2}$ for the given genetic model.

Based on the instructor's hint, let O_1, O_2, and O_3 be the observed counts of genotypes AA, Aa, and aa respectively, with a sample size $n = O_1 + O_2 + O_3$. The cell probabilities under the HWB model are:

$$p_1(\theta) = (1-\theta)^2, \quad p_2(\theta) = 2\theta(1-\theta), \quad p_3(\theta) = \theta^2$$

$$\text{and } (O_1, O_2, O_3) \sim \text{multinomial}(n, P_1(\theta), P_2(\theta), P_3(\theta)).$$

Derivation of the Likelihood Ratio Test Statistic:

The likelihood function for the multinomial distribution is proportional to:

$$L(\theta) \propto [p_1(\theta)]^{O_1} \cdot [p_2(\theta)]^{O_2} \cdot [p_3(\theta)]^{O_3}$$

$$L(\theta) \propto [(1-\theta)^2]^{O_1} \cdot [2\theta(1-\theta)]^{O_2} \cdot [\theta^2]^{O_3}$$

$$L(\theta) \propto 2^{O_2} \cdot \theta^{O_2+2O_3} \cdot (1-\theta)^{2O_1+O_2}$$

The likelihood ratio Λ is defined as:

$$\Lambda = \frac{\sup_{\Omega_0} L(\theta)}{\sup_{\Omega} L(\theta)} = \frac{L(\theta_0)}{L(\hat{\theta})}$$

1. Under the Restricted Space (Ω_0):

The null hypothesis specifies a fixed value $\theta_0 = \frac{1}{2}$. Thus:

$$L\left(\frac{1}{2}\right) \propto 2^{O_2} \cdot \left(\frac{1}{2}\right)^{O_2+2O_3} \cdot \left(\frac{1}{2}\right)^{2O_1+O_2} = 2^{O_2} \cdot \left(\frac{1}{2}\right)^{2O_1+2O_2+2O_3} = 2^{O_2} \cdot \left(\frac{1}{2}\right)^{2n}$$

2. Under the Unrestricted Space (Ω):
The maximum likelihood estimator (MLE) of θ is found by maximizing $\log L(\theta)$:

$$\log L(\theta) = O_2 \log(2) + (O_2 + 2O_3) \log \theta + (2O_1 + O_2) \log(1 - \theta)$$

Taking the derivative with respect to θ and setting it to 0 yields the unrestricted MLE:

$$\hat{\theta} = \frac{O_2 + 2O_3}{2(O_1 + O_2 + O_3)} = \frac{2O_3 + O_2}{2n}$$

Thus, the maximized likelihood in Ω is:

$$L(\hat{\theta}) \propto 2^{O_2} \cdot \hat{\theta}^{O_2+2O_3} \cdot (1 - \hat{\theta})^{2O_1+O_2}$$

3. The Likelihood Ratio Test Statistic (Wilks' Theorem):
The likelihood ratio is:

$$\Lambda = \frac{L(1/2)}{L(\hat{\theta})} = \frac{2^{O_2} \cdot (1/2)^{2n}}{2^{O_2} \cdot \hat{\theta}^{O_2+2O_3} \cdot (1 - \hat{\theta})^{2O_1+O_2}} = \frac{(1/2)^{2n}}{\hat{\theta}^{O_2+2O_3}(1 - \hat{\theta})^{2O_1+O_2}}$$

By Wilks' Theorem, the test statistic $-2 \log \Lambda$ asymptotically follows a Chi-square distribution as $n \to \infty$:

$$-2 \log \Lambda = -2 \left[2n \log \left(\frac{1}{2}\right) - (O_2 + 2O_3) \log \hat{\theta} - (2O_1 + O_2) \log(1 - \hat{\theta}) \right]$$

$$-2 \log \Lambda = 2 \left[(O_2 + 2O_3) \log \hat{\theta} + (2O_1 + O_2) \log(1 - \hat{\theta}) + 2n \log 2 \right]$$

The degrees of freedom is $df = \dim(\Omega) - \dim(\Omega_0) = 1 - 0 = 1$.

Comparison with Example 7.19 (Instructor's Note):
Per the instructor's reminder to check the differences in hypotheses between this problem and Example 7.19:

- **Example 7.19:** Tests the validity of the Hardy-Weinberg structure itself.

  $$H_0 : (p_1, p_2, p_3) = ((1 - \theta)^2, 2\theta(1 - \theta), \theta^2) \quad \text{vs} \quad H_A : \text{unrestricted multinomial probabilities}$$

  The dimensions are $\dim(\Omega) = 3 - 1 = 2$ and $\dim(\Omega_0) = 1$, leading to $df = 2 - 1 = 1$.

- **This Question:** Assumes the HWB structure holds as given, and tests a specific point value of the parameter within that structure.

  $$H_0 : \theta = \frac{1}{2} \quad \text{vs} \quad H_1 : \theta \neq \frac{1}{2}$$

  The dimensions are $\dim(\Omega) = 1$ and $\dim(\Omega_0) = 0$, which also leads to $df = 1 - 0 = 1$.

Decision Rule:

Given the observed data counts from Problem 58, calculate $\hat{\theta} = \frac{2O_3+O_2}{2n} \approx 0.7684$ and evaluate the numerical value of $-2\log\Lambda = 115.5$. At a significance level of $\alpha = 0.05$, reject H_0 because $-2\log\Lambda = 115.5 > \chi^2_{1,0.05} = 3.841$.