1. Ch9 #8

$H_0: \lambda = \lambda_0$ v.s. $H_a: \lambda > \lambda_0$

First. consider $H_0: \lambda = \lambda_0$ v.s. $H_a: \lambda = \lambda_a$. where $\lambda_a > \lambda_0$

⇒ ① The hypothesis testing is a simple to simple test

② By HW9 #7. the likelihood ratio test rejects H_0 if $\sum_{i=1}^n X_i$ is greater than c with significant level α. where c only depends on λ_0 and is irrelated to λ_a

⇒ By Neyman-Pearson lemma. any other test with significant level less than or equal to α has less power than that of likelihood ratio test. since c is irrelated to λ_a. the argument holds $\forall \lambda_a > \lambda_0$. Thus. the likelihood ratio test is the UMP test for $H_a: \lambda > \lambda_a$.

2. Ch9 #18.

$H_0: \lambda = \lambda_0$ v.s. $H_a: \lambda = \lambda_a$ where $\lambda_0 < \lambda_a$.

$$L(\lambda | x_1, \cdots, x_n) = f_{x_1, \cdots, x_n}(x_1, \cdots, x_n | \lambda) \overset{iid}{=} \prod_{i=1}^n \frac{1}{2} \lambda \exp\{-\lambda |x_i|\}$$

$$\Lambda = \frac{L(\lambda_0 | \underset{\sim}{x})}{L(\lambda_1 | \underset{\sim}{x})} = \frac{\prod_{i=1}^n \frac{1}{2} \lambda_0 \exp\{-\lambda_0 |x_i|\}}{\prod_{i=1}^n \frac{1}{2} \lambda_a \exp\{-\lambda_a |x_i|\}} = \left(\frac{\lambda_0}{\lambda_a}\right)^n \exp\{(\lambda_a - \lambda_0) \sum_{i=1}^n |x_i|\}$$

Λ is increasing w.r.t $\sum_{i=1}^n |x_i|$ since $\lambda_a - \lambda_0 > 0$

The likelihood ratio test rejects H_0 if $\Lambda < C$

⇒ Reject H_0 if $\sum_{i=1}^n |x_i| < C'$ where C' depends on the null distribution of $\sum_{i=1}^n |X_i|$ at λ_0 i.e. Gamma(n. λ_0) and is irrelated to λ_a. by (*) in next page.

① The hypothesis testing is a simple to simple test

② Reject H_0 if the likelihood ratio is smaller than c with significant level α. where c is irrelated to λ_a.

⇒ By Neyman-Pearson lemma. the test is UMP test for $H_a: \lambda > \lambda_a$

2. (*)

$F_{|X|}(k) = P(|X| \le k) = P(-k \le X \le k) = \int_{-k}^{k} \frac{1}{2}\lambda \exp\{-\lambda|x|\}dx = 2\int_0^k \frac{1}{2}\lambda \exp\{-\lambda x\}dx$

$= 1 - e^{-\lambda k}$

$\Rightarrow$ If $f_X(x) = \frac{1}{2}\lambda \exp\{-\lambda|x|\}$, $x \in \mathbb{R}$, $|X| \sim \exp(\lambda)$

$\Rightarrow \sum_{i=1}^{n} |X_i| \sim \Gamma(n, \lambda)$

$\Rightarrow 2\lambda \sum_{i=1}^{n} |X_i| \sim \Gamma(n, \frac{1}{2}) \sim \chi^2_{2n}$

$\Rightarrow P(2\lambda \sum_{i=1}^{n} |X_i| < \chi^2_{2n}(1-\alpha)) = \alpha$

$\Rightarrow P(\sum_{i=1}^{n} |X_i| < \frac{\chi^2_{2n}(1-\alpha)}{2\lambda}) = \alpha$

$\Rightarrow C' = \frac{\chi^2_{2n}(1-\alpha)}{2\lambda} \underset{H_0}{\text{under}} \frac{\chi^2_{2n}(1-\alpha)}{2\lambda_0}$

$\Rightarrow C'$ is irrelated to λ_a.

3. $H_0: f = f_0 \sim \text{Poisson}(1)$ v.s. $H_a: f = f_1 \sim \text{Geo}(\frac{1}{2})$

since the dimension of parameter is 0, the hypothesis is simple. By Neyman-pearson lemma, the likelihood ratio test is an MP test.

$$\Lambda = \frac{f_0(x_1, \cdots, x_n)}{f_1(x_1, \cdots, x_n)} = \frac{\prod_{i=1}^{n} \frac{e^{-1}}{x_i!}}{\prod_{i=1}^{n} (\frac{1}{2})^{x_i}} = e^{-n} \prod_{i=1}^{n} \frac{2^{x_i}}{x_i!}$$

Reject region: $\Lambda < C$

Random test function $\phi(x) = \begin{cases} 1 & \Lambda < C \\ 0 & \Lambda > C \\ \gamma & \Lambda = C \end{cases}$, C and γ is defined by

$P(R.R|H_0) = E(\phi(X)|H_0) = P(\Lambda < C|H_0) + \gamma P(\Lambda = C|H_0) = \alpha$

since $\gamma > 0$ and $P(\Lambda < C|H_0)$ need to be as closed as possible to α but smaller than or equal to $\alpha \Rightarrow \gamma = \frac{\alpha - P(\Lambda < C|H_0)}{P(\Lambda = C|H_0)}$

4.

$H_0: \theta \geq \theta_0 \quad H_a: \theta < \theta_0$

$$f_{X_1 \cdots X_n}(x_1 \cdots x_n) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}}$$

$$= \theta^{-n} e^{-\frac{\sum_{i=1}^{n} x_i}{\theta}} = L(\theta)$$

is a one-parameter exponential family

with $T(X) = \sum_{i=1}^{n} X_i$ and $c(\theta) = -\frac{1}{\theta}$ being an increasing function of θ

By the answer to the question 7.10 in LNp.4-20

the UMP test function is

$$\phi(X) = \begin{cases} 1 & \text{if } T(X) = \sum_{i=1}^{n} X_i \leq C \\ 0 & \text{if } T(X) = \sum_{i=1}^{n} X_i > C \end{cases}$$

To determine C

$$P(T(X) \leq C \mid \theta = \theta_0) = \alpha$$

$$\sum_{i=1}^{n} X_i \overset{\theta=\theta_0}{\sim} \Gamma(n, 1/\theta_0)$$

$$\Rightarrow \frac{2}{\theta_0} \sum_{i=1}^{n} X_i \sim \chi^2_{2n}$$

$$P\left(\frac{2}{\theta_0} \sum_{i=1}^{n} X_i \leq \frac{2}{\theta_0} C \mid \theta = \theta_0\right) = \alpha$$

$$\frac{2C}{\theta_0} = \chi^2_{2n,1-\alpha}$$

$$C = \frac{\theta_0}{2} \chi^2_{2n,1-\alpha}$$

5. $H_0: p=0.25 \quad H_A: p\neq 0.25$

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3)=\prod_{i=1}^{3} p^{x_i}(1-p)^{1-x_i}$$
$$=\left(\frac{p}{1-p}\right)^{\sum_{i=1}^{3}x_i}(1-p)^3$$
$$=\exp\left\{\left(\sum_{i=1}^{3}x_i\right)\log\frac{p}{1-p}\right\}(1-p)^3$$

is an one-parameter exponential family.

$\log\frac{p}{1-p}$ is an increasing function w.r.t p

let $Y=\sum_{i=1}^{3}X_i$ and test function

$$\phi(x)=\begin{cases}1 & Y<C_1 \text{ or } Y>C_2\\ \gamma_1 & Y=C_1\\ \gamma_2 & Y=C_2\\ 0 & \text{o.w.}\end{cases}$$

to determine C_1 C_2 γ_1 γ_2

$$\alpha=E[\phi(x)\mid H_0]$$

$$\alpha E[Y\mid H_0]=E[\phi(x)Y\mid H_0]$$

under $H_0: p=0.25$, $Y\sim Bin(3,0.25)$. 故

$P(Y\le 0)=P(Y=0)=\binom{3}{0}(0.25)^0(0.75)^3=0.42>0.05$ So $C_1=0$

$P(Y\ge 3)=P(Y=3)=\binom{3}{3}(0.25)^3(0.75)^0=0.02<0.05$

$P(Y\ge 2)=P(Y=2)+P(Y=3)=\binom{3}{2}(0.25)^2(0.75)^1+0.02=0.14+0.02>0.05$

$C_2=2$ or 3

If $C_2 = 2$

$\alpha = 0.05 = E[\phi(X)|H_0]$

$= \gamma_1 P(Y=0) + \gamma_2 P(Y=2) + P(Y=3)$

$= 0.42\gamma_1 + 0.14\gamma_2 + 0.02$ — (1)

$0.05(3-0.25) = E(\phi(X)|H_0)E(Y|H_0) = E[Y\phi(X)|H_0]$

$= 0\cdot\gamma_1 P(Y=0) + 2\gamma_2 P(Y=2) + 3\cdot P(Y=3)$

$= 0.28\gamma_2 + 0.06$ — (2)

用(1)(2)求解後得 $\gamma_2 < 0$ (contradict)

If $C_2 = 3$

$\alpha = 0.05 = E[\phi(X)|H_0]$

$= \gamma_1 P(Y=0) + \gamma_2 P(Y=3)$

$= 0.42\gamma_1 + 0.02\gamma_2$ — (3)

$0.05\cdot(3-0.25) = E(\phi(X)|H_0)E(Y|H_0) = E[Y\phi(X)|H_0]$

$= 0\cdot\gamma_1 P(Y=0) + 3\gamma_2 P(Y=3)$

$= 0.06\gamma_2$ — (4)

用(3)(4)求解得 $\gamma_2 = 0.625 > 0$

$\gamma_1 = \dfrac{0.05 - 0.02\cdot 0.625}{0.42} \approx 0.0893 > 0$

$\begin{cases} C_1 = 0 \\ C_2 = 3 \\ \gamma_1 = 0.0893 \\ \gamma_2 = 0.625 \end{cases}$ by thm such test $\phi(X)$ is the UMPU test