G_1 **Theorem** (best *constant* predictor under MSE). a constant function of x; minimum

$$E_{X,Y}(Y-c)^2 = E_Y(Y-c)^2 \geq E_Y[Y-E_Y(Y)]^2 = Var_Y(Y)$$

The equality holds if and only if $c=E_Y(Y)$. only need to know μ_Y

Proof. $h(Y)=(Y-c)^2$: a function of Y

$E_{X,Y}[h(Y)] = E_Y[h(Y)]$

$\| $ LNp.8-22

$E_Y E_{X|Y}[h(Y)|Y]$, $h(Y)$

Thm in LNp.5-19

$$E_Y(Y-c)^2 = Var_Y(Y) + (\mu_Y - c)^2 \geq Var_Y(Y)$$

LNp.8-24; cf. LNp.8-24

12/12

G_3 **Theorem** (best predictor under MSE). minimum

$$E_{X,Y}[Y-g(X)]^2 \geq E_{X,Y}[Y-E_{Y|X}(Y|X)]^2 = E_X[Var_{Y|X}(Y|X)]$$

(*)

The equality holds if and only if $g(x)=E_{Y|X}(Y|x)$. cf. (LNp.8-21)

a function of X only

Proof.

$$\begin{aligned} E_{X,Y}[Y-g(X)]^2 &= E_{X,Y}\{[Y-E_{Y|X}(Y|X)]+[E_{Y|X}(Y|X)-g(X)]\}^2 \\ &= E_{X,Y}[Y-E_{Y|X}(Y|X)]^2 + \underbrace{E_X}_{E_{X,Y}}[E_{Y|X}(Y|X)-g(X)]^2 \\ &\quad + 2\cdot E_{X,Y}\{[Y-E_{Y|X}(Y|X)][E_{Y|X}(Y|X)-g(X)]\} \quad \leftarrow =0 \\ &= E_{X,Y}[Y-E_{Y|X}(Y|X)]^2 + E_X[E_{Y|X}(Y|X)-g(X)]^2 \\ &\geq E_{X,Y}[Y-E_{Y|X}(Y|X)]^2 \end{aligned}$$

a function of X only

last "="

$=0$ iff $g(x)=E_{Y|X}(Y|X)$

where the last "=" comes from

$$E_{X,Y}\{[Y-E_{Y|X}(Y|X)][E_{Y|X}(Y|X)-g(X)]\} \quad h(X,Y)$$

$$= E_X E_{Y|X}\{[Y-E_{Y|X}(Y|X)][E_{Y|X}(Y|X)-g(X)] \mid X\}$$

By the law of total expectation (LNp.8-22) $E_{XY}[h(X,Y)] = E_X E_{Y|X}[h(X,Y)|X]$

this is a constant when conditioned on X

$$= E_X\{[E_{Y|X}(Y|X)-g(X)]\, E_{Y|X}[Y-E_{Y|X}(Y|X)|X]\} = 0.$$

$= E_{Y|X}(Y|X) - E_{Y|X}(Y|X)$

important concept: mean is best predictor under MSE

Furthermore, (for (*) in LNp.8-27)

$$E_{X,Y}[Y-E_{Y|X}(Y|X)]^2 = E_X E_{Y|X}\{[Y-E_{Y|X}(Y|X)]^2|X\} = E_X[Var_{Y|X}(Y|X)]$$

➢ Some notes for the best predictor in G_3

best in G_1: $E_Y(Y)$; cf.

- $E_{Y|X}(Y|x)$ is the best predictor of Y based on X, in the sense of mean square prediction error. intuition ← check the graph in LNp.8-20
- Its calculation requires to know the joint distribution of X and Y, or at least $E_{Y|X}(Y|x)$
- $E_{Y|X}(Y|x)$ is called the regression function of Y on X 迴歸

G2

Theorem (best *linear* predictor under MSE). $-1 \le \rho_{XY} \le 1$

$$E_{X,Y}[Y-(a+bX)]^2 \ge E_{X,Y}\left\{Y - \left[\mu_Y + \rho_{XY}\frac{\sigma_Y}{\sigma_X}(X-\mu_X)\right]\right\}^2$$

minimum $= \sigma_Y^2(1-\rho_{XY}^2)$

The equality holds if and only if $a=\mu_Y - b\mu_X$ and $b=\rho_{XY}\sigma_Y/\sigma_X$ unit = ?

Proof. $E_{X,Y}(Y-a-bX)^2$ $R(X,Y) \equiv Z$

$\because Var(Z) = E(Z^2) - [E(Z)]^2$

$$= Var_{X,Y}(Y-a-bX) + [E_{X,Y}(Y-a-bX)]^2$$

$$= Var_{X,Y}(Y-bX) + (\mu_Y - a - b\mu_X)^2$$

$$\boxed{\ge} \; Var_{X,Y}(Y-bX) \quad (\Rightarrow \text{setting } a=\mu_Y - b\mu_X)$$

Thm in LNp.8-13

$$= \sigma_Y^2 + b^2\sigma_X^2 - 2b\sigma_{XY}$$

$$= \sigma_X^2\left(b^2 - 2b\frac{\sigma_{XY}}{\sigma_X^2} + \frac{\sigma_{XY}^2}{\sigma_X^4}\right) + \sigma_Y^2 - \frac{\sigma_{XY}^2}{\sigma_X^2}$$

$$= \sigma_X^2\left(b - \frac{\sigma_{XY}}{\sigma_X^2}\right)^2 + \sigma_Y^2(1-\rho_{XY}^2)$$

$$\boxed{\ge} \; \sigma_Y^2(1-\rho_{XY}^2) \quad \left(\Rightarrow \text{setting } b = \frac{\sigma_{XY}}{\sigma_X^2} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}\times\frac{\sigma_Y}{\sigma_X} = \rho_{XY}\frac{\sigma_Y}{\sigma_X}\right)$$

➢ Some notes for the best *linear* predictor in G_2

- $E_{Y|X}(Y|x) = \mu_Y + (\rho_{XY}\sigma_Y/\sigma_X)(x-\mu_X)$ if (X, Y) is distributed as bivariate normal. linear regression analysis

best in G_3 best in G_2

- Its calculation requires to know the means, variances, and covariance of X and Y. $\xleftrightarrow{\text{cf.}}$ best in G_3, G_1 Which one require more information ?

MSE = 0 if $\rho_{XY} = \pm 1$ MSE $= \sigma_Y^2$ if $\rho_{XY} = 0$

- $\sigma_Y^2(1-\rho_{XY}^2)$ is small if ρ_{XY} is close to +1 or −1, and large if ρ_{XY} is close to 0. intuition? check the correlation plots in LNp.8-9 more information better predictor

- A comparison of these minimum MSEs

$\because G_1 \subset G_2 \subset G_3$

- ⊳ $\min_{a,b} E_{X,Y}[Y-(a+bX)]^2 \le \min_c E_{X,Y}(Y-c)^2$ and the equality holds if and only if $\rho_{XY} = 0$.
- ⊳ $\min_g E_{X,Y}[Y-g(X)]^2 \le \min_{a,b} E_{X,Y}[Y-(a+bX)]^2$ and the equality holds if and only if $E_{Y|X}(Y|x) = \mu_Y + (\rho_{XY}\sigma_Y/\sigma_X)(x-\mu_X)$.

❖ **Reading**: textbook, Sec 7.6

Moment Generating Function

- Definition (Moment and Central Moment). If a random variable X has a cdf F_X, then

$$\mu_k \equiv E(X^k) = \int_{-\infty}^{\infty} x^k \, dF_X(x), \quad k = 1, 2, 3, \ldots,$$

are called the k^{th} *moments* of X provided that the integral converges absolutely, and

$$\mu_k' \equiv E[(X - \underbrace{\mu_X}_{\text{a constant}})^k] = \int_{-\infty}^{\infty} (x - \mu_X)^k \, dF_X(x), \quad k = 2, 3, \ldots,$$

are called k^{th} *moment about the mean* μ_X or *central moment* of X provided that the integral converges absolutely.

➢ Some notes.

- $\mu_k' = E[(X - \mu_X)^k] = E\left[\sum_{i=0}^{k} \binom{k}{i} (-\mu_X)^{k-i} X^i\right]$

  $= \sum_{i=0}^{k} \binom{k}{i} (-\mu_X)^{k-i} E(X^i) = \sum_{i=0}^{k} \binom{k}{i} (-\mu_X)^{k-i} \mu_i.$

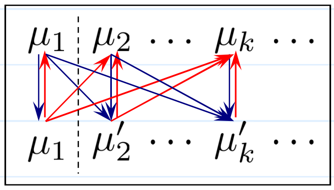
  $\boxed{\mu_0 = E(X^0) = 1}$

- $\mu_k = E(X^k) = E\{[(X - \mu_X) + \mu_X]^k\}$

  $= \sum_{i=0}^{k} \binom{k}{i} (\mu_X)^{k-i} E[(X - \mu_X)^i]$

  $= \sum_{i=0}^{k} \binom{k}{i} (\mu_X)^{k-i} \mu_i'.$

  $\boxed{\mu_0' = 1, \ \mu_1' = 0}$

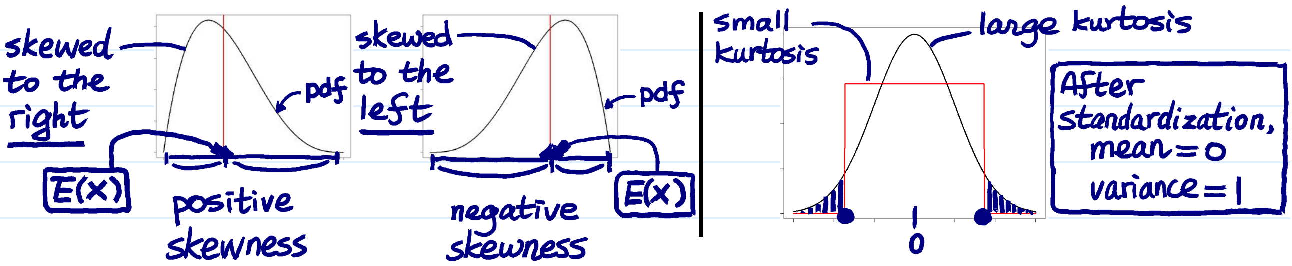


- In particular,

$$E(X) = \mu_X = \mu_1, \quad \text{and,}$$
$$Var(X) = \sigma_X^2 = \mu_2' = \mu_2 - \mu_1^2. = E(x^2) - [E(x)]^2$$

$\boxed{\text{Recall. mean, var, cov, cor}}$ $\boxed{\text{defined by expectation}}$

- The (central) moments give a lot of useful information about the distribution in addition to mean and variance, e.g.,
  - Skewness (a measure of the asymmetry): $\mu_3'/\sigma^3 . = E\left(\frac{X-\mu}{\sigma}\right)^3$
  - Kurtosis (a measure of the "heavy tails"): $\mu_4'/\sigma^4 . = E\left(\frac{X-\mu}{\sigma}\right)^4$

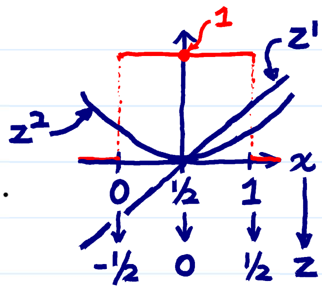


➢ Example (Uniform). If $X \sim \text{Uniform}(0, 1)$, then

$$\mu_k = \int_0^1 x^k \, dx = \frac{1}{k+1},$$

therefore, $\mu_X = \mu_1 = 1/2, \quad \text{and,}$

$\mu_2' \rightarrow \sigma_X^2 = \mu_2 - \mu_1^2 = 1/3 - (1/2)^2 = 1/12.$



And, $\mu_k' = \int_0^1 (x - 1/2)^k \, dx = \int_{-1/2}^{1/2} z^k \, dz$

$\boxed{\text{skewness} = 0, \ \text{kurtosis} = 1.8}$

$$= \frac{1}{k+1}\left[(1/2)^{k+1} - (-1/2)^{k+1}\right] = \begin{cases} 0, & k \text{ is odd,} \\ \frac{1}{(k+1)2^k}, & k \text{ is even.} \end{cases}$$

- Recall. How to characterize a distribution?

  (1) pdf/pmf, (2) cdf, (3) mgf

- Definition (Moment Generating Function). If X is a random variable with the cdf F_X, then

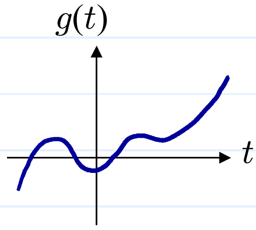
$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\, dF_X(x),$$

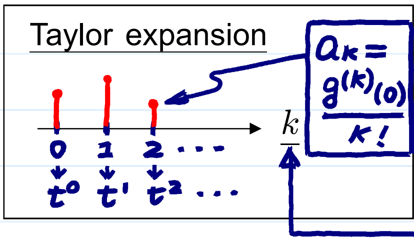
$= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$ for continuous case

is called the *moment generating function* (mgf) of X provided that the integral converges absolutely in some non-degenerate interval of t.

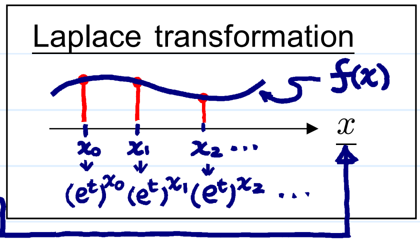
Q: how to express a function?

$$g(t) \boxed{=} \sum_{k=0}^{\infty} a_k t^k \qquad g(t) \boxed{=} \int_{\mathbb{R}} f(x)\,(e^t)^x\, dx$$





cf.



➢ Some Notes.

- The mgf is a function of the variable t.
- The mgf may only exist for some particular values of t. — i.e., not all $t \in \mathbb{R}$
- $M_X(t)$ always exists at $t=0$ and $M_X(0)=1$ ← Thm in LNp.8-36

➢ Example.

- If X is a discrete r.v. taking on values x_i's with probability p_i's, $i=1, 2, 3, \ldots$, then

$$M_X(t) = E(e^{tX}) = \sum_{i=1}^{\infty} e^{tx_i} p_i.$$

- If $X \sim$ Poisson(λ), then for $-\infty < t < \infty$,

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \times \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\left(e^{\lambda e^t}\right) \sum_{x=0}^{\infty} \underbrace{\frac{e^{-(\lambda e^t)}(\lambda e^t)^x}{x!}}_{\text{pmf of Poisson}(\lambda e^t)} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}.$$

- If $X \sim$ exponential(λ), then for $t < \lambda$,

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} \times \lambda e^{-\lambda x}\, dx$$

$$= \lambda\left(\frac{1}{\lambda - t}\right) \int_0^{\infty} \underbrace{(\lambda - t)\, e^{-(\lambda - t)x}}_{\text{pdf of exponential}(\lambda - t)} dx = \frac{\lambda}{\lambda - t},$$

STO

This must be > 0

and $M_X(t)$ does not exist for $t \geq \lambda$.

- A list of some mgfs (exercise)
  - If $X \sim$ binomial(n, p), — use binomial expansion (LNp.5-23)

$$M_X(t) = (1 - p + pe^t)^n, \text{ for } t < -\log(1-p).$$

$r=1$, geometric distribution

- If $X \sim$ negative binomial(r, p),

use negative binomial expansion (LNp.5-29)

$$M_X(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r, \text{ for } t < -\log(1-p).$$

$\alpha=1$, exponential distribution

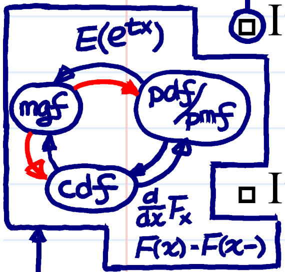
- If $X \sim$ uniform(α, β), $M_X(t) = \dfrac{e^{\beta t} - e^{\alpha t}}{t(\beta-\alpha)}$.

- If $X \sim$ gamma(α, λ), use STO

$$M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^\alpha, \text{ for } t < \lambda.$$

use $e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}$ & STO

- If $X \sim$ beta(α, β), $M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r}\right) \frac{t^k}{k!}$.



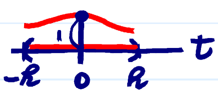
- If $X \sim$ normal(μ, σ^2), $M_X(t) = e^{\mu t + (\sigma^2/2)t^2}$. use STO

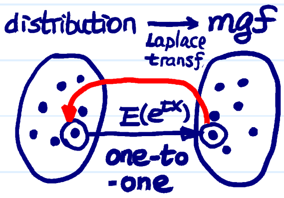
Theorem (Uniqueness Theorem). Suppose that the mgfs $M_X(t)$ and $M_Y(t)$ of random variables X and Y exist for all $|t|<h$ for some $h>0$. If

$$M_X(t) = M_Y(t),$$

for $|t|<h$, then

i.e. an open interval containing zero





does not mean X=Y

$$F_X(z) = F_Y(z)$$

X, Y have same dist.

for all $z \in \mathbb{R}$, where F_X and F_Y are the cdfs of X and Y, respectively.

Proof. Skipped (by the uniqueness theorem of Laplace transform.)

➢ Application of the uniqueness theorem

- When a mgf exists for all $|t|<h$ for some $h>0$, there is a *unique* distribution corresponding to that mgf.
- This allows us to use mgfs to find distributions of *transformed* random variables in some cases.
- This technique is most commonly used for *linear combinations* of independent random variables $X_1, \ldots, X_n$.

Find dist. of $X_1+\cdots+X_n$ (check the Thms in LNp.8-38)

➢ Example. If $M_X(t) = p_1 e^{a_1 t} + \cdots + p_k e^{a_k t}$, where $p_1+\cdots+p_k=1$, then X is a discrete r.v. and its pmf is

by uniqueness Thm & mgf in LNp.8-34

$$p_X(x) = \begin{cases} p_i, & \text{for } x = a_i,\ i = 1, \ldots, k, \\ 0, & \text{otherwise.} \end{cases}$$

- **Theorem** (Moments and MGF). If $M_X(t)$ exists for $|t|<h$ for some $h>0$, then

can take derivative at $t=0$

$$M_X(0)=1,$$

related to the coefficients in the Taylor expansion of $M_X(t)$

know all moments ⇒ know dist.

and, kth derivative

$$M_X^{(k)}(0) = \mu_k, \quad k = 1, 2, 3, \ldots$$

This explains why it's called moment generating function.

Proof. First, $M_X(0) = \int_{-\infty}^{\infty} e^{0\cdot x}\, dF_X(x) = \underbrace{\int_{-\infty}^{\infty} 1\, dF_X(x)}_{= F_X(x)\big|_{-\infty}^{\infty} =} = 1.$

$$M_X{}'(0) = \frac{d}{dt} M_X(t)\Big|_{t=0} = \left[\frac{d}{dt}\int_{-\infty}^{\infty} e^{tx}\, dF_X(x)\right]\Big|_{t=0}$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{dt} e^{tx}\Big|_{t=0}\right) dF_X(x) = \int_{-\infty}^{\infty} \left(xe^{tx}|_{t=0}\right) dF_X(x)$$

$$= \int_{-\infty}^{\infty} x\cdot 1\, dF_X(x) = E_X(X) = \mu_1.$$

$$\cdots = \cdots$$

$$M_X^{(k)}(0) = \frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \left[\frac{d^k}{dt^k}\int_{-\infty}^{\infty} e^{tx}\, dF_X(x)\right]\Big|_{t=0}$$

$$= \int_{-\infty}^{\infty} \left(\frac{d^k}{dt^k} e^{tx}\Big|_{t=0}\right) dF_X(x) = \int_{-\infty}^{\infty} \left(x^k e^{tx}|_{t=0}\right) dF_X(x)$$

$$= \int_{-\infty}^{\infty} x^k\cdot 1\, dF_X(x) = E_X(X^k) = \mu_k.$$

➢ Example. If $X \sim$ exponential(λ), then $M_X(t) = \frac{\lambda}{\lambda - t}$. ← $\boxed{\text{LNp.8-34}}$

Because $M_X^{(k)}(t) = \frac{k!\,\lambda}{(\lambda-t)^{k+1}}$,

we get $\mu_k = M_X^{(k)}(0) = \frac{k!}{\lambda^k}$. → Then, can use kth moments to obtain mean, variance, skewness, kurtosis, ..., kth central moments, ...