➢Example. If a sample of n balls is drawn without replacement from a box containing R red balls, W white balls, and $N-R-W$ blue balls. Let

If with replacement, $(X, Y, n-X-Y) \sim$ multinomial

$n-X-Y$: # of blue balls

X = # of red balls in the sample,

Y = # of white balls in the sample,

generalization of hypergeometric ← without or with replacement (similar to the generalization from binomial to multinomial)

then, the joint pmf of (X, Y) is

$$p_{X,Y}(x,y) = \frac{\binom{R}{x}\binom{W}{y}\binom{N-R-W}{n-x-y}}{\binom{N}{n}},$$

$x \in \{0, 1, \cdots, R\}$
$y \in \{0, 1, \cdots, W\}$
$0 \le x+y \le n$

Find $E_Y(Y)$.

Sol. Because $Y|X=x \sim$ hypergeometric$(n-x, N-R, W)$,

$$g(x) \equiv E_{Y|X}(Y|X=x) = (n-x)[W/(N-R)].$$

LNp.5-42

Because $X \sim$ hypergeometric$(n, N, R) \Rightarrow E_X(X) = n(R/N)$, and then

$$\begin{aligned} E_Y(Y) &= E_X[E_{Y|X}(Y|X)] = E_X[g(X)] \\ &= E_X\left[(n-X)\frac{W}{N-R}\right] = \frac{W}{N-R}[n - E_X(X)] \\ &= \frac{W}{N-R}\left(n - n\frac{R}{N}\right) = n\frac{W}{N}. \end{aligned}$$

$E_Y E_{X|Y} = E_{X,Y}$

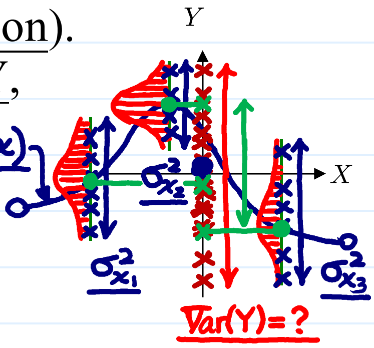
Note that $Y \sim$ hypergeometric$(n, N, W) \Rightarrow E_Y(Y) = n(W/N)$.

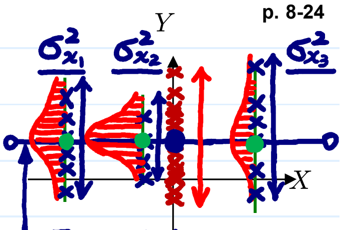
⊙ Theorem (Variance Decomposition). For two random vectors $\mathbf{X}$ and $\mathbf{Y}$,

$$\begin{aligned} Var_{\mathbf{Y}}(Y_i) &= Var_{Y_i}(Y_i) \\ &= Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})] \\ &\quad + E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]. \end{aligned}$$

"between group" variation

"within group" variation





$E_{Y|X}(Y|x)$: a constant function of x.

The concept behind the Thm leads to "Analysis of Variance (ANOVA)" in statistics.

Proof. $Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{x}) = E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{x}) - [E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{x})]^2$,

and, $E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$

$$= E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\}.$$

Note. $Var(Z) = E(Z-\mu)^2 = E(Z^2) - [E(Z)]^2$

Also, $Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$ $\equiv g(X) = Z$ cf. same

$$= E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} - \{E_{\mathbf{X}}[\underbrace{E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})}_{g(X)}]\}^2.$$

Now, $Var_{\mathbf{Y}}(Y_i) = E_{\mathbf{Y}}(Y_i^2) - [E_{\mathbf{Y}}(Y_i)]^2$

$E_{X,Y}$ $E_{X,Y}$

Law of Total Expectation (LNp.8-22)
Recall. $E_Y(h(Y)) = E_Y E_{X|Y}(h(Y)) = E_{X,Y}(h(Y))$

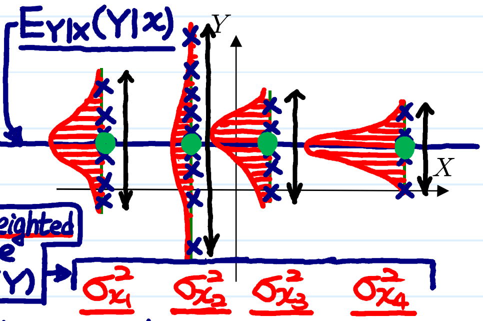
$$\begin{aligned} &= E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - \{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]\}^2 \\ &= E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} \\ &\quad + E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} - \{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]\}^2 \\ &= E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})] + Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]. \end{aligned}$$

➢Corollary.

- $Var_{\mathbf{Y}}(Y_i) \geq E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$ and the equality holds if and only if

$$E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X}) = E_{\mathbf{Y}}(Y_i)$$

with probability one.



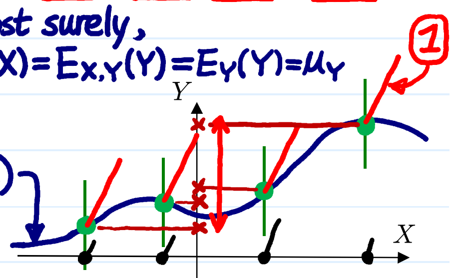
$Var_X[\underset{g(X)}{\underset{|||}{E_{Y|X}(Y|X)}}]=0 \Rightarrow g(x)=E_{Y|X}(Y|x)$ is a constant over x almost surely, and the constant is $E_X[g(X)]=E_XE_{Y|X}(Y|X)=E_{X,Y}(Y)=E_Y(Y)=\mu_Y$

$Y=g(X)$
$Var_Y(Y) = Var_X(g(X))$

- $Var_{\mathbf{Y}}(Y_i) \geq Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$ and the equality hold if and only if $g(x) \overset{|||}{=} E_{Y|X}(Y|x)$

$h(X) = Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})=0$ ($\Rightarrow Y_i = E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})$)

with probability one. $E_X[Var_{Y|X}(Y|X)]=0$, $\overset{|||}{h(X)} \geq 0$



$\sigma^2_{x_1} = \sigma^2_{x_2} = \sigma^2_{x_3} = \sigma^2_{x_4}$
$\| \quad \| \quad \| \quad \|$
$0 \quad 0 \quad 0 \quad 0$

❖ **Reading**: textbook, Sec 7.5

Conditional Expectation and Prediction

can be a random vector

- Problem formulation: predicting the value of a r.v. Y on the basis of the observed value of a r.v. X ← fixed random

strong relation or association between X & Y ⇒ good prediction

➢Data: X and Y (example?)

X	身高	雨量	地點
Y	體重	米產量	房價

➢Statistical modeling: assigning (X, Y) a (known) joint distribution (cdf $F(x, y)$, pdf $f(x, y)$, or pmf $p(x, y)$)

If unknown, a statistical problem

➢Objective: predicting Y by using a function of X, i.e.,

how the distribution can help?

When $X=x$, use $g(x)$ to predict value of Y → $g(X)$ ← predictor ← e.g., for example in LNp.8-21, g transform year into cm

➢Predictor: considering the following three groups of g's

(i) $G_1 = \{g(x) : g(x) = c, \text{ where } c \in \mathbb{R}\}$ ← not use the information of X

(ii) $G_2 = \{g(x) : g(x) = a + bx, \text{ where } a, b \in \mathbb{R}\}$

(iii) $G_3 = \{g(x) : g \text{ is an arbitrary function}\}$

Note. $G_1 \subset G_2 \subset G_3$. ← best in G_2 is no worse than best in G_1; best in G_3 " " " best in G_2

➢Question: Within each group, what is the "best" predictor?

how to choose c in G_1? " " " a, b in G_2? " " " g in G_3?

➢Criterion: minimizing mean square error

meaning? → $\text{MSE} \equiv E_{X,Y}\{[Y - g(X)]^2\}$

Why taking expectation over X? true value error predicted value

G1 Theorem (best *constant* predictor under MSE).

$$E_{X,Y}(Y-c)^2 = E_Y(Y-c)^2 \geq E_Y[Y - \underbrace{E_Y(Y)}_{\text{a constant function of } x}]^2 = \underbrace{Var_Y(Y)}_{\text{minimum}}$$

The equality holds if and only if $c = E_Y(Y)$. ← only need to know μ_Y

Proof. $h(Y)=(Y-c)^2$: a function of Y

$E_{X,Y}[h(Y)] = E_Y[h(Y)]$ (LNp.8-22)

$\| $

$E_Y E_{X|Y}[h(Y)|Y]$, $E_{X|Y}[h(Y)|Y] = h(Y)$

Thm in LNp.5-19:

$$E_Y(Y-c)^2 = Var_Y(Y) + (\mu_Y - c)^2 \geq Var_Y(Y)$$

cf. LNp.8-24

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G3 Theorem (best predictor under MSE).

$$E_{X,Y}[Y-g(X)]^2 \geq E_{X,Y}[Y - E_{Y|X}(Y|X)]^2 \overset{(*)}{=} \underbrace{E_X[Var_{Y|X}(Y|X)]}_{\text{minimum}}$$

LNp.8-24

The equality holds if and only if $g(x) = E_{Y|X}(Y|x)$. ← a function of X only; cf. (LNp.8-21)

Proof. $E_{X,Y}[Y-g(X)]^2$

$$= E_{X,Y}\{[Y - E_{Y|X}(Y|X)] + [E_{Y|X}(Y|X) - g(X)]\}^2$$

$$= E_{X,Y}[Y - E_{Y|X}(Y|X)]^2 + \underbrace{E_X}_{E_{X,Y}}[E_{Y|X}(Y|X) - g(X)]^2$$ (a function of X only)

$$+ 2\cdot \underbrace{E_{X,Y}\{[Y-E_{Y|X}(Y|X)][E_{Y|X}(Y|X)-g(X)]\}}_{=0}$$

last "="

$$= E_{X,Y}[Y - E_{Y|X}(Y|X)]^2 + E_X[E_{Y|X}(Y|X) - g(X)]^2$$

$$\geq E_{X,Y}[Y - E_{Y|X}(Y|X)]^2$$

$=0$ iff $g(X) = E_{Y|X}(Y|X)$