- Corollary. Suppose that $X_1, \ldots, X_n$ are uncorrelated and have same mean μ and variance σ^2. Let

$$S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}{n-1},$$

then $E(S^2)=\sigma^2$.

Proof.

$$\begin{aligned}(n-1)S^2 &= \sum_{i=1}^{n}(X_i-\overline{X}_n)^2 \\ &= \sum_{i=1}^{n}[(X_i-\mu)-(\overline{X}_n-\mu)]^2 \\ &= \left[\sum_{i=1}^{n}(X_i-\mu)^2\right]+\left[\sum_{i=1}^{n}(\overline{X}_n-\mu)^2\right] \\ &\quad -2(\overline{X}_n-\mu)\left[\sum_{i=1}^{n}(X_i-\mu)\right] \\ &= \left[\sum_{i=1}^{n}(X_i-\mu)^2\right]+n(\overline{X}_n-\mu)^2-2n(\overline{X}_n-\mu)^2 \\ &= \left[\sum_{i=1}^{n}(X_i-\mu)^2\right]-n(\overline{X}_n-\mu)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}(n-1)E(S^2) &= \left\{\sum_{i=1}^{n}E\left[(X_i-\mu)^2\right]\right\}-nE\left[(\overline{X}_n-\mu)^2\right] \\ &= n\sigma^2 - nVar(\overline{X}_n) = (n-1)\sigma^2.\end{aligned}$$

- Note. The previous three corollaries also hold if $X_1, \ldots, X_n$ are "uncorrelated" is replaced by "independent."

Theorem (ρ of linear transformation).

$$Cor(a_0+a_1X, b_0+b_1Y)=\text{sign}(a_1b_1)\times Cor(X,Y),$$

and

$$|Cor(a_0+a_1X, b_0+b_1Y)|=|Cor(X,Y)|,$$

i.e., $|\rho_{XY}|$ is invariant under location and scale changes.

Proof. Let $S=a_0+a_1X$ and $T=b_0+b_1Y$, then

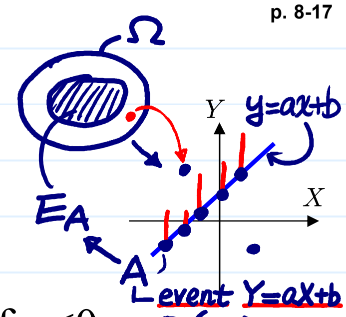
$$Cov(S,T)=Cov(a_0+a_1X, b_0+b_1Y)=a_1b_1Cov(X,Y),$$

$$Var(S)=a_1^2\,Var(X), \quad \text{and} \quad Var(T)=b_1^2\,Var(Y).$$

Therefore,

$$\rho_{ST}=\frac{Cov(S,T)}{\sigma_S\sigma_T}=\frac{a_1b_1Cov(X,Y)}{|a_1||b_1|\sigma_X\sigma_Y}=\frac{a_1b_1}{|a_1b_1|}\rho_{XY}.$$

$\frac{cov(X,Y)}{\sigma_X \sigma_Y}$ ➢ Theorem (some properties of ρ). (cf. → Cauchy-Schwarz inequality)



$= \boxed{0 \le |\rho_{XY}| \le 1}$ ⇔ (1) $-1 \le \rho_{XY} \le 1$. ($\Leftrightarrow |Cov(X,Y)| \le \sigma_X \sigma_Y$)

ρ is unit-free

(2) $\rho_{XY} = \pm 1$ if and only if there exist $a, b \in \mathbb{R}$

such that $P(Y=aX+b)=1$. ← $Y=aX+b$ almost surely

(3) Furthermore, $\rho_{XY}=1$, if $a>0$ and $\rho_{XY}=-1$, if $a<0$.

$P(E_A)=1$

$P(E_A^c)=0$

Proof of (1).

柯西不等式 → Cauchy-Schwarz inequality
$\underline{u}=(u_1,\cdots,u_n)$, $\underline{v}=(v_1,\cdots,v_n)$
$|\sum_i u_i v_i| = |\langle \underline{u}, \underline{v} \rangle| \le \|\underline{u}\| \cdot \|\underline{v}\| = \sqrt{\sum_i u_i^2} \cdot \sqrt{\sum_i v_i^2}$
$Y(\omega) \times \sqrt{P(\omega)}$, $X(\omega) \times \sqrt{P(\omega)}$ ← $\sum_i \to \sum_{\omega \in \Omega}$

$$0 \le Var\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right)$$

Thm in LNp.8-13 →

$$= Var\left(\frac{X}{\sigma_X}\right) + Var\left(\frac{Y}{\sigma_Y}\right) + 2Cov\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right)$$

(σ_X, σ_Y: constants)

$$= \frac{Var(X)}{\sigma_X^2} + \frac{Var(Y)}{\sigma_Y^2} + 2\frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$= 1 + 1 + 2\rho_{XY} \quad \Rightarrow \rho_{XY} \ge -1.$$

$|E(XY)| \le \sqrt{E(X^2)}\sqrt{E(Y^2)}$

Similarly,

$|Cov(X,Y)| = |E[(X-\mu_X)(Y-\mu_Y)]| \le \sqrt{E[(X-\mu_X)^2]}\sqrt{E[(Y-\mu_Y)^2]} = \sigma_X \sigma_Y$

$$0 \le Var\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = 1 + 1 - 2\rho_{XY} \Rightarrow \rho_{XY} \le 1.$$

Proof of (2) and (3). We see from the proof of (1),

$$\rho_{XY} = 1 \Leftrightarrow Var\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = 0,$$

$Var(Z)=0 \Leftrightarrow Z=c$ almost surely, for a constant c →

$$\Leftrightarrow \quad P\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y} = c\right) = 1,$$

where c is a constant.

$$\Leftrightarrow \quad P\left(Y = \frac{\sigma_Y}{\sigma_X} X + c\sigma_Y\right) = 1.$$

($\frac{\sigma_Y}{\sigma_X} > 0$)

dist. unknown

fixed

- different transformation g ⇒ different information
- These expections are called parameters in statistics
- parameters can be estimated by r.v.'s (transformation of data), e.g., $\bar{X}_n \xrightarrow{e} \mu$, $S^2 \xrightarrow{e} \sigma^2$

Similarly, $\rho_{XY} = -1 \quad \Leftrightarrow \quad P\left(Y = -\frac{\sigma_Y}{\sigma_X} X + c\sigma_Y\right) = 1.$ ($-\frac{\sigma_Y}{\sigma_X} < 0$)

- **Q**: How to use expectations to (roughly) characterize the distribution of random variables $X_1, \ldots, X_n$?

  ➢ $g(X_1, \ldots, X_n) = X_i \Rightarrow E[g(\mathbf{X})] = \mu_{X_i}$: *mean* of X_i. ← g: 1st order polynomials of $X_1,\cdots,X_n$

  ➢ $g(X_1, \ldots, X_n) = (X_i - \mu_{X_i})^2 \Rightarrow E[g(\mathbf{X})] = \sigma^2_{X_i}$: *variance* of X_i.

  ➢ $g(X_1, \ldots, X_n) = (X_i - \mu_{X_i})(X_j - \mu_{X_j})$ for $i \ne j$

  $\Rightarrow E[g(\mathbf{X})] = \sigma_{X_i X_j}$: *covariance* of X_i and X_j.

  g: 2nd order polynomials of $X_1,\cdots,X_n$

  ➢ $g(X_1, \ldots, X_n) = [(X_i - \mu_{X_i})/\sigma_{X_i}][(X_j - \mu_{X_j})/\sigma_{X_j}]$ for $i \ne j$

  $\Rightarrow E[g(\mathbf{X})] = \rho_{X_i X_j}$: *correlation coefficient* of X_i and X_j.

  ➢ Notes. $\mu_{X_i}, \sigma^2_{X_i}, \sigma_{X_i X_j}, \rho_{X_i X_j}$ are constants, not random

❖ **Reading**: textbook, Sec 7.1, 7.2, 7.4, 7.9

Recall. Conditional dist. LN p. 7-51~59

Conditional Expectation

$\mathbf{X}\in\mathbb{R}^n, \mathbf{Y}\in\mathbb{R}^m$

- Recall. $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ or $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ is a pmf/pdf for $\mathbf{y}$ ($\mathbf{y}$: random, $\mathbf{x}$: fixed).
- Definition. For random vectors $\mathbf{X}$ and $\mathbf{Y}$, the *conditional expectation* of $Z=h(\mathbf{Y})$ given $\mathbf{X}=\mathbf{x}$, where $h: \mathbb{R}^m\to\mathbb{R}^1$, is

平均: $R(Y)$ 權重: $P_{Y|X}(y|x)$

$$E_{\mathbf{Y}|\mathbf{X}}\left(h(\mathbf{Y})\,\middle|\,\mathbf{X}=\mathbf{x}\right)=\sum_{\mathbf{y}\in\mathcal{Y}} h(\mathbf{y})\,p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}),$$

in the discrete case, or,

$$= \Sigma_z\, z\, P_{Z|X}(z|x)$$

(1)=(2) in LN p.8-1

平均: $R(Y)$ 權重: $f_{Y|X}(y|x)$

$$E_{\mathbf{Y}|\mathbf{X}}\left(h(\mathbf{Y})\,\middle|\,\mathbf{X}=\mathbf{x}\right)=\int_{\mathbb{R}^m} h(\mathbf{y})\,f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\,d\mathbf{y},$$

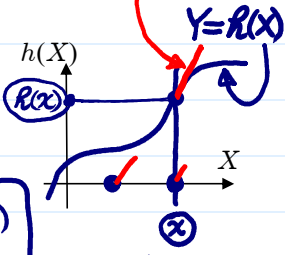
in the continuous case,

$$= \int_z z\, f_{Z|X}(z|x)\,dz$$

(3)=(4) in LN p.8-1

provided that the sum or integral converges absolutely.

cond. prob = 1



➢ Some Notes.

- $E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})\mid \mathbf{X}=\mathbf{x})$: a function of $\mathbf{x}$ and free of $\mathbf{Y}$.
- $E_{\mathbf{Y}|\mathbf{X}}[\,h(\mathbf{X})\mid \mathbf{X}=\mathbf{x}\,]=h(\mathbf{x})$.

$E_{Y|x}[R(X,Y)|X=x]=\int R(x,y) f_{Y|x}(y|x)\,dy$

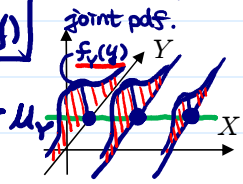
$\because f_{Y|x}(y|x)=f_Y(y)$, $P_{Y|x}(y|x)=P_Y(y)$

- If $\mathbf{X}$ and $\mathbf{Y}$ are independent, then

$$E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})|\mathbf{X}=\mathbf{x})=E_{\mathbf{Y}}[h(\mathbf{Y})].$$

a constant line of x

μ_Y



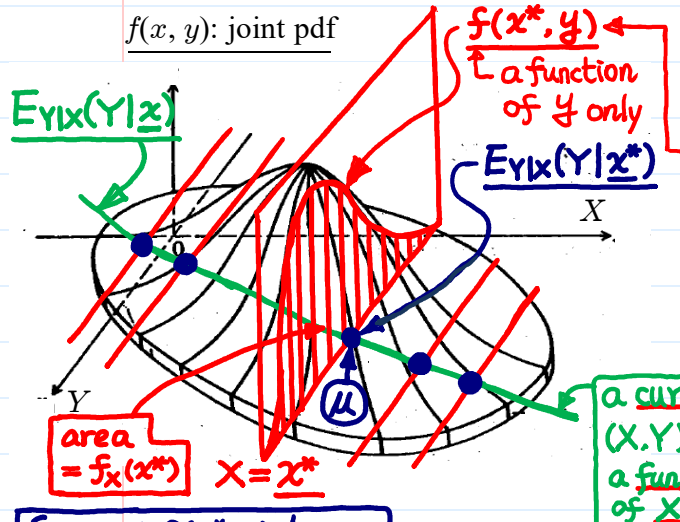
- Let $g(\mathbf{x})=E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}=\mathbf{x}]$, where $g:\mathbb{R}^n\to\mathbb{R}^1$, then we write

cf. $E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})|\mathbf{X})$

after cf. before

when $\mathbf{x}$ in g is replaced by $\mathbf{X}$ (a fixed value replaced by a r.v.).

  - Notice that $g(\mathbf{X})$ is a random variable.



cf.

$$f_X(x^*)=\int_{-\infty}^{\infty} f(x^*,y)\,dy \overset{?}{=} 1.$$

➢ $f(x, y)$: a joint pdf.

➢ Fix x^*, is $f(x^*, y)$ a pdf of y? i.e.,

➢ $f_{Y|X}(y|x^*)=f(x^*, y)/f_X(x^*)$ is a pdf of y since

$$\frac{\int_{-\infty}^{\infty} f(x^*,y)\,dy}{f_X(x^*)}=1.$$

center of gravity

$\int (y-\mu)\,f(x^*,y)\,dy=0$

$\Rightarrow \int y f(x^*,y)\,dy=\mu\int f(x^*,y)\,dy=\mu f_X(x^*)$

$\Rightarrow \mu=\int y\,f(x^*,y)/f_X(x^*)\,dy=E_{Y|x}(Y|x^*)$

$\Rightarrow f_{Y|x}(y|x^*)$ & $f(x^*,y)$ have same center of gravity

$R(Y)=Y$

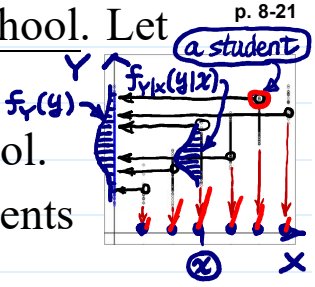
➢ $E_{Y|X}(Y|x^*)$: mean of $f_{Y|X}(y|x^*)$.

➢ Do it for any $x=x^*$, and get a function of x $\Rightarrow E_{Y|X}(Y|x)$

➢ Example. Sample a student from an elementary school. Let

X=age (unit: year), Y=height (unit: cm)

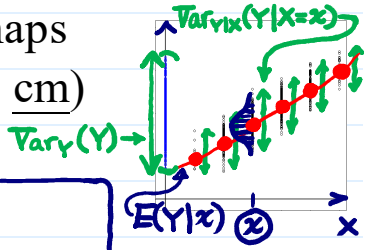
of the student. **Population**: all students of the school.



Q: What's the source of their randomness?

- $Y|X=x$: a random variable (unit: cm) that represents the height distribution of students with age=x.
- $g(x)=E_{Y|X}(Y|X=x)$ or $E_{Y|X}(Y|x)$: a *function* maps from age (unit: year) to average height (unit: cm) of students with age=x.

cf. after / before

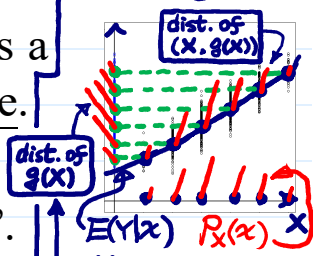


Note. $E_{Y|X}(Y|x)$ is not a random variable.

- $g(X)=E_{Y|X}(Y|X)$: a random variable because it is a function of age X, where X is a random variable.

$R(Y)=[Y-g(x)]^2$, $E_{Y|x}[R(Y)|x]$

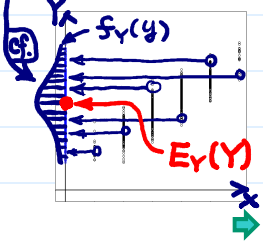
$g(x)$: 1-to-1 in this case $\Rightarrow P(E_{Y|x}(Y|X)=E_{Y|x}(Y|x))=P(X=x)$



Note. $g(X)=E_{Y|X}(Y|X)$ is height, its unit is "cm".

- $Var_{Y|X}(Y|X=x)$ & $Var_{Y|X}(Y|X)$ defined similarly.
  - $E_Y(Y)$: average height of ***all*** students;
  - $Var_Y(Y)$: variation of height of ***all*** students.

cf. cf.



- Theorem (Law of Total Expectation). For two random vectors $\mathbf{X}$ ($\in\mathbb{R}^m$) and $\mathbf{Y}$ ($\in\mathbb{R}^n$),

$$E_{\mathbf{X}}\{E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}]\}=E_{\mathbf{Y}}[h(\mathbf{Y})].$$

$E_{X,Y}[R(Y)]=E_Y E_{X|Y}$; $E_{X|Y}[R(Y)]=R(Y)$

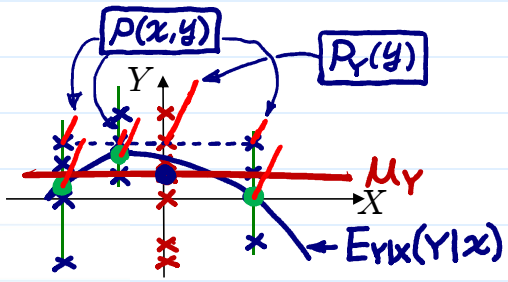
use the example in LNp.8-21 to realize the meaning of these terms.

In particular, let $h(\mathbf{Y})=Y_i$, we have

$$E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]=E_{\mathbf{Y}}(Y_i).$$

Y_i & $E_{Y|X}(Y_i|X)$ have same mean

$g(X)$; $E_{Y_i}(Y_i)=E_{X,Y}(Y_i)$



Proof.

(only prove it for the continuous case)

$$E_{\mathbf{X}}\{E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}]\}$$

$$=\int_{\mathbb{R}^m} E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})|\mathbf{x})\, f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

$$=\int_{\mathbb{R}^m}\left[\int_{\mathbb{R}^n} h(\mathbf{y}) f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y}\right] f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

interchange dy & dx: dydx → dxdy

$$=\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} h(\mathbf{y})\frac{f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{f_{\mathbf{X}}(\mathbf{x})} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}d\mathbf{y}$$

$E_{X,Y}[R(Y)]$

$$=\int_{\mathbb{R}^n} h(\mathbf{y})\left[\int_{\mathbb{R}^m} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})\, d\mathbf{x}\right] d\mathbf{y}$$

multiplication law (LNp.7-55): $f_{X|Y}(x|y)\times f_Y(y)\,dxdy \Rightarrow E_{X,Y}=E_Y E_{X|Y}$

$$=\int_{\mathbb{R}^n} h(\mathbf{y}) f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}$$

$$=E_{\mathbf{Y}}[h(\mathbf{Y})].$$

E_Y: ① dx + ② dy; $E_X E_{Y|X}$: ① dy + ② dx

generalization:

$E_{X,Y}[R(X,Y)]$
$=E_X E_{Y|X}[R(X,Y)|X]$
$=E_Y E_{X|Y}[R(X,Y)|Y]$