property on only 2 r.v.'s

cf.

- n pairwisely indep. r.v.'s
- n mutually indep. r.v.'s

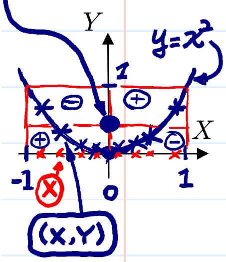
- Corollary. If X and Y are independent, then $Cov(X, Y)=0$, i.e., X and Y are uncorrelated. $\rho_{XY}=0$

Proof. When X, Y are independent,

$$E(XY)=E(X)E(Y)=\mu_X\mu_Y.$$

by Corollary in LNp.8-6

$(\mu_X, \mu_Y)=(0, 1/3)$



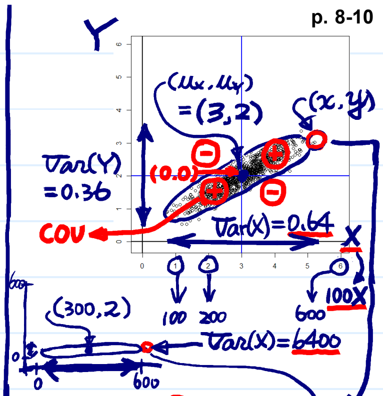
□ However, the converse statement is *not* necessarily true. → "uncorrelated" is a weaker condition than "indep."

(e.g., let X~Uniform(−1, 1) and $Y=X^2$, then

$$Cov(X, Y)=0, \leftarrow E(XY)=E(X^3)=\int_{-1}^{1} x^3\,\tfrac{1}{2}dx=0 \qquad E(X)E(Y)=0\cdot(1/3)=0$$

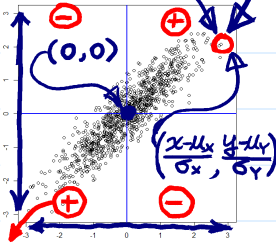
but X and Y are *not* independent).

Q: What's the dist. of $Y|X=x$? $P(Y=y|X=x)=\begin{cases}1, & \text{if } y=x^2\\ 0, & \text{o.w.}\end{cases}$



$Y'=\frac{Y-\mu_Y}{\sigma_Y}$

Var(Y')=1



cov = cor

$X'=\frac{X-\mu_X}{\sigma_X}$

Var(X')=1

Q: Why is cor unit-free?

X', Y': mean = 0 variance = 1

→ cov(X', Y') = cor(X', Y') = cor(X, Y)

standardization (標準化, LNp.6-33)

● Corollary.

$$\rho_{XY}=E\left[\underbrace{\left(\frac{X-\mu_X}{\sigma_X}\right)}_{X'}\underbrace{\left(\frac{Y-\mu_Y}{\sigma_Y}\right)}_{Y'}\right].$$

$E[(X-\mu_X)(Y-\mu_Y)]/\sigma_X\sigma_Y$

Proof. By definition.

Note. Y is a function of X. Zero correlation does not mean No relationship. cf. ✽ in LNp 8-9.

➢ Example. If $(X_1, \ldots, X_m)$ ~ Multinomial$(n, m, p_1, \ldots, p_m)$, then

$$Cov(X_i, X_j) = -np_ip_j, \quad \text{for } 1 \le i \ne j \le m.$$

- Because $(X_1, X_2, X_3+\cdots+X_m)$ ~ Multinomial$(n, 3, p_1, p_2, p_3+\cdots+p_m)$, and

$$X_3+\cdots+X_m = n-X_1-X_2, \qquad p_3+\cdots+p_m = 1-p_1-p_2,$$

we have

$0\le x_1+x_2\le n$, $0\le x_1\le n$, $0\le x_2\le n$

$$E(X_1X_2) = \sum x_1x_2\binom{n}{x_1,x_2,n-x_1-x_2}p_1^{x_1}p_2^{x_2}(1-p_1-p_2)^{n-x_1-x_2}$$

$2\le x_1+x_2\le n$, $1\le x_1\le n-1$, $1\le x_2\le n-1$

$$= \sum \cancel{x_1}\cancel{x_2}\frac{n!}{\underbrace{\cancel{x_1}!}_{(x_1-1)}\underbrace{\cancel{x_2}!}_{(x_2-1)}(n-x_1-x_2)!}p_1^{x_1}p_2^{x_2}(1-p_1-p_2)^{n-x_1-x_2}$$

$$= n(n-1)p_1p_2\left[\sum \frac{(n-2)!}{(x_1-1)!(x_2-1)!(n-x_1-x_2)!} \times (p_1)^{x_1-1}(p_2)^{x_2-1}(1-p_1-p_2)^{n-x_1-x_2}\right]$$

$0\le y_1+y_2\le n-2$, $0\le y_1\le n-2$, $0\le y_2\le n-2$

y_1 y_2 $(n-2)-y_1-y_2$

joint pmf of Multinomial$(n-2, 3, p_1, p_2, 1-p_1-p_2)$

$$= n(n-1)p_1p_2.$$

- WLOG, we can get $E(X_iX_j) = n(n-1)p_ip_j$, for $i \ne j$.

Therefore,

$$Cov(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j) = n(n-1)p_ip_j - (np_i)(np_j) = -np_ip_j.$$

- And, for $i \neq j$,

Note. $X_1+X_2+X_3+\cdots+X_m=n$ → Why negative

$$\underline{Cor(X_i, X_j)} = \frac{-\cancel{n}\sqrt{p_i}\sqrt{p_j}}{\sqrt{\cancel{n}\cancel{p_i}(1-p_i)}\sqrt{\cancel{n}\cancel{p_j}(1-p_j)}} = -\sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}}.$$

Q: When larger? When smaller?
Hint. $\frac{p}{1-p}\uparrow$ as $p\uparrow$
Note. $0 \le p_i + p_j \le 1$

⊙ Cov & Cor for Sums of Random Variables

cf

➢ Notation. In the following, let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be r.v.'s and $-\infty < a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m < \infty$ are constants.

➢ Recall. $E(a_0+a_1X_1+\cdots+a_nX_n) = a_0+a_1E(X_1)+\cdots+a_nE(X_n)$. ← exchange of E & Σ

LNp.8-4

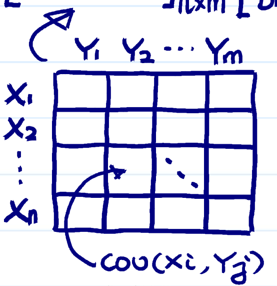
➢ Theorem (covariance of two sums).

$$Cov(\underline{a_0} + a_1X_1 + \cdots + a_nX_n, \underline{b_0} + b_1Y_1 + \cdots + b_mY_m)$$

$$= \sum_{i=1}^n \sum_{j=1}^m a_ib_jCov(X_i, Y_j) \rightarrow [a_1 \cdots a_n]\begin{bmatrix} cov(X_i,Y_j) \end{bmatrix}_{n\times m}\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

a_0, b_0 are gone.

Proof. Let $S = a_0+a_1X_1+\cdots+a_nX_n$, and $T= b_0+b_1Y_1+\cdots+b_mY_m$, then



$$S - \overbrace{E(S)}^{a_0+a_1\mu_{X_1}+\cdots+a_n\mu_{X_n}} = \sum_{i=1}^n a_i(X_i - \mu_{X_i}),$$

$$T - \overbrace{E(T)}^{b_0+b_1\mu_{Y_1}+\cdots+b_m\mu_{Y_m}} = \sum_{j=1}^m b_j(Y_j - \mu_{Y_j}),$$

$$[S - E(S)][T - E(T)] = \sum_{i=1}^n \sum_{j=1}^m a_ib_j(X_i - \mu_{X_i})(Y_j - \mu_{Y_j}).$$

Therefore, $Cov(S,T) = E\{[S - E(S)][T - E(T)]\}$

$$= \sum_{i=1}^n \sum_{j=1}^m a_ib_jE[(X_i - \mu_{X_i})(Y_j - \mu_{Y_j})]$$

$$= \sum_{i=1}^n \sum_{j=1}^m a_ib_jCov(X_i, Y_j).$$

cf. mean of sum (LNp.8-4)

➢ Theorem (variance of sum). → $Cov(a_0+a_1X_1+\cdots+a_nX_n, a_0+a_1X_1+\cdots+a_nX_n)$

presents some useful information about any 2 of the n r.v.'s ← covariance matrix of $X_1, \ldots, X_n$

$$Var(\underline{a_0} + \overbrace{a_1X_1}^{Y_1=} + \cdots + \overbrace{a_nX_n}^{Y_n=})$$

gone

$$= \sum_{i=1}^n \sum_{j=1}^n a_ia_jCov(X_i, X_j) \longrightarrow [a_1,\cdots,a_n]\begin{bmatrix} cov(X_i,X_j) \end{bmatrix}_{n\times n}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Var(X_i) ↘ (diagonal)

symmetric matrix

$$= \underbrace{\sum_{i=1}^n a_i^2Var(X_i)}_{=Var(Y_i)}$$

$$+2\underbrace{\sum_{1\le i<j\le n} a_ia_jCov(X_i, X_j)}_{=Cov(Y_i,Y_j)}.$$

Proof. $Cov(X_i, X_i)=Var(X_i)$ and $Cov(X_i, X_j)= Cov(X_j, X_i)$.

cf.

▪ Corollary. If $X_1, \ldots, X_n$ are uncorrelated, then

$cor(X_i,X_j)=0$, $\forall i,j$ $i\neq j$, i.e., $[cov(X_i,X_j)]$ is a diagonal matrix

$$Var(\underbrace{a_0}_{gone} + \overbrace{a_1X_1}^{Y_1} + \cdots + \overbrace{a_nX_n}^{Y_n}) = \sum_{i=1}^n \overbrace{a_i^2Var(X_i)}^{Var(Y_i)}. \quad \boxed{\tfrac{1}{n^2}\times n\sigma^2}$$

exchange of Var & Σ

▪ Corollary. If $X_1, \ldots, X_n$ are uncorrelated and

$$Var(X_1)=\cdots = Var(X_n) \equiv \sigma^2 < \infty,$$ ← Variance exists

$\frac{X_1+\cdots+X_n}{n} \rightarrow a_1=\cdots=a_n=\frac{1}{n}$

then $Var(\overline{X}_n) = \boxed{\sigma^2/n}$ ≈ 0 when $n\uparrow\infty$ cf. ★ in LNp.8-5

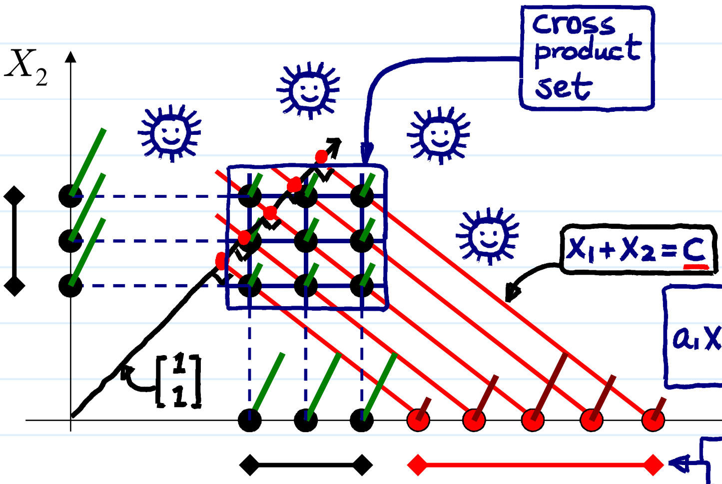
→ Law of Large Number

i.e. $\overline{X}_n \approx \underline{C_n}$ when n large enough. a constant → $C_n = ?\ E(\overline{X}_n)=?$

Independent

$Cov(X_1, X_2) = 0$

$\sigma_{X_1}^2 = \sigma_{X_2}^2$



$$Var(X_1 + X_2) = Var(X_1) + Var(X_2)$$

$$Var(X_1 - X_2) = Var(X_1) + Var(X_2)$$

$a_1X_1+\cdots+a_nX_n = \underbrace{[a_1,\cdots,a_n]}_{\underset{\sim}{a}^T} \underbrace{\begin{bmatrix}X_1\\ \vdots \\ X_n\end{bmatrix}}_{\underset{\sim}{x}}$

$\underset{\sim}{a}^T\underset{\sim}{x} = |\underset{\sim}{a}| \cdot \frac{\underset{\sim}{a}^T}{|\underset{\sim}{a}|} \underset{\sim}{x}$

smaller variance

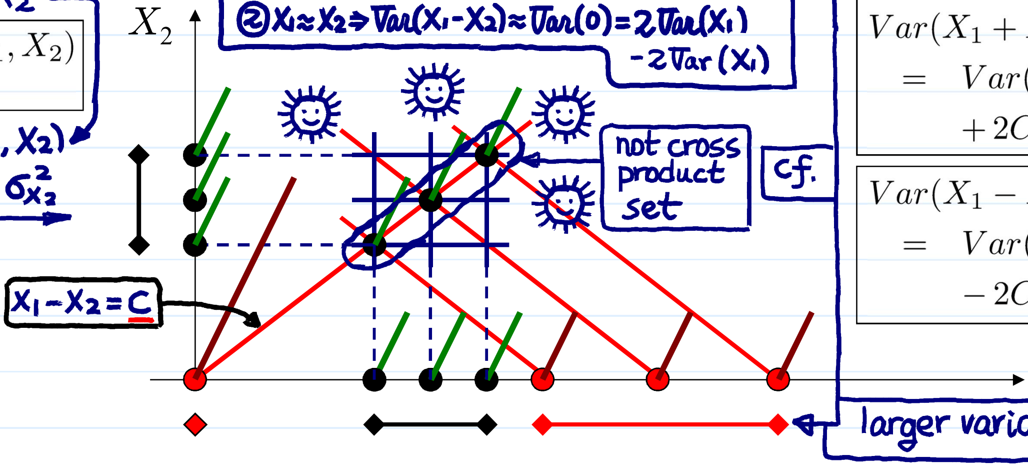
same marginals
different joints

Why? ① $X_1 \approx X_2 \Rightarrow Var(X_1+X_2) \approx Var(2X_1) = 4Var(X_1)$
② $X_1 \approx X_2 \Rightarrow Var(X_1-X_2) \approx Var(0) = 2Var(X_1)$
$-2Var(X_1)$

$X_1 = X_2$

$Cov(X_1, X_2) > 0$

$cov(X_1, X_2) = \sigma_{X_1}^2 = \sigma_{X_2}^2$



cf.

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$$

$$Var(X_1 - X_2) = Var(X_1) + Var(X_2) - 2Cov(X_1, X_2)$$

larger variance