Expectation

$Y=g(X):\mathbb{R}^1\to\mathbb{R}^1$ (LNp.5-13~19, LNp.6-13~16)

- Recall. Expectation for univariate random variable.

- Theorem. For random variables $\mathbf{X}=(X_1,\ldots,X_n)$ with joint pmf $p_{\mathbf{X}}$/pdf $f_{\mathbf{X}}$, the *expectation* of a univariate random variable Y, where

$$Y=g(X_1,\ldots,X_n),\qquad g:\mathbb{R}^n\to\mathbb{R}^1, \leftarrow \text{why?}$$

is (discrete; by definition)

$$E(Y) \stackrel{\text{by definition}}{=} \sum_{y\in\mathcal{Y}} y\, p_Y(y) \tag{1}$$

no need to use $p_Y(y)$

$$= \sum_{\mathbf{x}=(x_1,\ldots,x_n)\in\mathcal{X}} g(x_1,\ldots,x_n)\, p_{\mathbf{X}}(x_1,\ldots,x_n) \tag{2}$$

$$\equiv E[g(X_1,\ldots,X_n)]$$

$\equiv$: defined as; $\mathcal{X}$: range of $\mathbf{X}$

if $X_1,\ldots,X_n$ are discrete and the sum converges absolutely, or ($\sum_{\mathbf{x}\in\mathcal{X}}|g(\mathbf{x})|\,p_{\mathbf{X}}(\mathbf{x})<\infty$)

$$E(Y) \stackrel{\text{by definition}}{=} \int_{-\infty}^{\infty} y\, f_Y(y)\, dy \tag{3}$$

no need to use $f_Y(y)$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\ldots,x_n) f_{\mathbf{X}}(x_1,\ldots,x_n)\, dx_1\cdots dx_n \tag{4}$$

$$\equiv E[g(X_1,\ldots,X_n)]$$

$\equiv$: defined as; $\int_{\mathbb{R}^n}$

if Y and $X_1,\ldots,X_n$ are continuous and the integrals converges absolutely. $\leftarrow \int_{\mathbb{R}^n}|g(\mathbf{x})|\,f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x}<\infty$

Proof. Like the univariate case. $\leftarrow$ proof in LNp.5-15 & LNp.6-16 ($g:\mathbb{R}^1\to\mathbb{R}^1$)

Q: What if Y is discrete and $X_1,\ldots,X_n$ are continuous? $\leftarrow$ e.g. $Y=g(\mathbf{X})=\begin{cases}1, & \text{if } \mathbf{X}\in A\subset\mathbb{R}^n\\ 0, & \text{otherwise}\end{cases}$

(1)=(4)

➢ Notation.

- Shorthand notation. Combine (1) and (3) by writing

Note: cdf is defined for any r.v.'s

$$E(Y)=\int_{-\infty}^{\infty} y\, dF_Y(y) = \begin{cases}\sum_{y\in\mathcal{Y}} y\, p_Y(y), & \text{for discrete case,}\\ \int_{-\infty}^{\infty} y\, f_Y(y)\, dy, & \text{for continuous case,}\end{cases}$$

F_Y: cdf of Y

$dF_Y(y)\approx F_Y(y)-F_Y(y-)$

Note: $dF_Y(y)/dy=f_Y(y)$ $\Rightarrow dF_Y(y)=f_Y(y)\,dy$

$dF_Y(y)\approx F_Y(y+\frac{dy}{2})-F_Y(y-\frac{dy}{2})=P(y-\frac{dy}{2}<Y\le y+\frac{dy}{2})$ and let $dy\to 0$

and combine (2) and (4) by writing

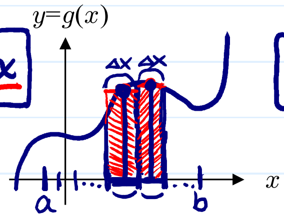
$$E[g(\mathbf{X})]=\int_{\mathbb{R}^n} g(\mathbf{x})\, dF_{\mathbf{X}}(\mathbf{x}) = \begin{cases}\sum_{\mathbf{x}\in\mathcal{X}} g(\mathbf{x})\, p_{\mathbf{X}}(\mathbf{x}), & \text{for discrete case,}\\ \int_{\mathbb{R}^n} g(\mathbf{x})\, f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}, & \text{for continuous case.}\end{cases}$$

$F_{\mathbf{X}}$: joint cdf of $\mathbf{X}$

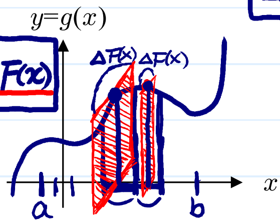
- Riemann-Stieltjes Integral.

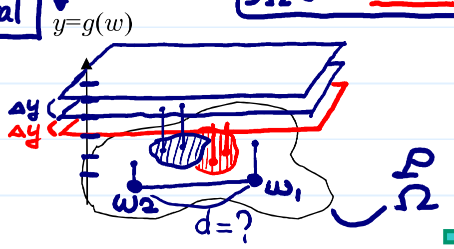
Lebesgue Integral; $Y=g(\omega)$: a r.v.; $\int_\Omega g(\omega)\,dP(\omega)$

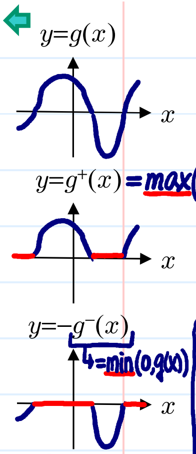
$\int_a^b g(x)\,dx$



$\int_a^b g(x)\,dF(x)$







For example, for non-negative g, and non-decreasing, right-continuous F, e.g. a cdf

weight of $(x_{i-1}, x_i]$ ↔ $P((x_{i-1}, x_i])$

$$\int_a^b g(x)\,dF(x) = \lim \sum_{i=1}^n g(x_i)[F(x_i) - F(x_{i-1})].$$

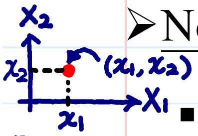
where the limit is taken over all $a=x_0<x_1<\cdots<x_n=b$ as $n\to\infty$ and $\max_{i=1,\ldots,n}(x_i - x_{i-1}) \to 0$.

[Recall. The integral of g over (a, b] is defined as cf.

$$\int_a^b g(x)\,dx = \lim \sum_{i=1}^n g(x_i)(x_i - x_{i-1}).]$$

length of $(x_{i-1}, x_i]$

$\int g(x)dF(x) = \int g^+(x)dF(x) + \int g^-(x)dF(x)$



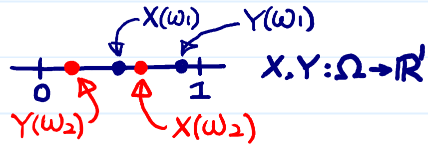
➢ Note. For their calculation, it is enough to know the marginal dist.

- $g(X_1, \ldots, X_n) = X_i \Rightarrow E[g(X_1, \ldots, X_n)] = E(X_i) \equiv \mu_{X_i}$. a projection
- $g(X_1, \ldots, X_n) = (X_i - \mu_{X_i})^2 \Rightarrow E[g(X_1, \ldots, X_n)] = Var(X_i) \equiv \sigma^2_{X_i}$. $= E[(X_i - \mu_{X_i})^2] =$ This is a fixed value, not random.

➢ Example (Average distance between two points). Suppose that

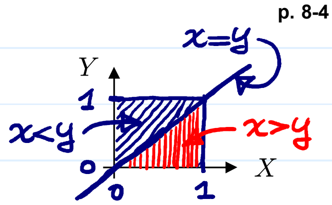
jointly distributed X, Y are i.i.d. ~ Uniform(0, 1).

Let $D = |X - Y|$. Find $E(D)$. $g(X, Y): \mathbb{R}^2 \to \mathbb{R}^1$



- The joint pdf of (X, Y) is

$$f(x,y) = \begin{cases} 1, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$



Note. not necessary to derive the pdf of D.

- $E(D) = \int_0^1\int_0^1 |x-y|\,dydx = \int_0^1 \left[\int_0^x (x-y)\,dy + \int_x^1 (y-x)\,dy\right] dx$

$$= \int_0^1 \left[-\tfrac{1}{2}(y-x)^2\Big|_{y=0}^{x} + \tfrac{1}{2}(y-x)^2\Big|_{y=x}^{1} \right] dx$$

$$= \int_0^1 \tfrac{1}{2}\left[x^2 + (1-x)^2\right] dx = \tfrac{1}{6}\left[x^3 - (1-x)^3\right]\Big|_{x=0}^1 = \tfrac{1}{3}.$$

negative a_i's ⇒ difference

⊙ Theorem (Mean of Sum). For jointly distributed r.v.'s $X_1, \ldots, X_n$ — a special form of g.

and constants $-\infty < a_0, a_1, \ldots, a_n < \infty$,

$$E(a_0 + \underbrace{a_1 X_1}_{Y_1} + \cdots + \underbrace{a_n X_n}_{Y_n}) = a_0 + a_1 \underbrace{E(X_1)}_{=E(Y_1)} + \cdots + a_n \underbrace{E(X_n)}_{=E(Y_n)}.$$

$Y = g(X_1, \ldots, X_n)$

Note. They can be any r.v.'s with finite means (no additional condition is required)

Proof. $E(a_0 + a_1X_1 + \cdots + a_nX_n)$ exchange of E & Σ — Why?

$$= \int_{\mathbb{R}^n} (a_0 + a_1x_1 + \cdots + a_nx_n)\,dF_{\mathbf{X}}(\mathbf{x})$$

$$= \int_{\mathbb{R}^n} a_0\,dF_{\mathbf{X}}(\mathbf{x}) + a_1\int_{\mathbb{R}^n} x_1\,dF_{\mathbf{X}}(\mathbf{x}) + \cdots + a_n \int_{\mathbb{R}^n} x_n\,dF_{\mathbf{X}}(\mathbf{x})$$

$$= a_0 + a_1E(X_1) + \cdots + a_nE(X_n).$$

Interchange of integration & summation ("=" guaranteed by absolute convergence)

$g_1(X_1, \ldots, X_n) = X_1$

$g_n(X_1, \ldots, X_n) = X_n$

only need the marginal dist. of $X_1, \ldots, X_n$, not necessary to know joint dist.

➢ Corollary. Suppose that $\mu = E(X_1) = \cdots = E(X_n)$. Let

e.g. $X_1, \ldots, X_n$ are identically distributed

$g(X_1, \cdots, X_n)$ a r.v.

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n},$$ $\to a_0 = 0,\ a_1 = a_2 = \cdots = a_n = \frac{1}{n}$

then, $E(\bar{X}_n) = \mu$. cf. definition of mean $= \underline{\underline{E}}(\underline{\underline{X}}) = \mu$

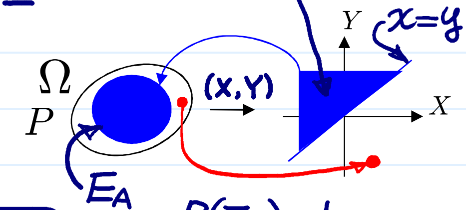
➢ Corollary. If X and Y are r.v.'s with finite means and

e.g. X:出生時身高 Y:10歲時身高

$$P(X \le Y) = 1,$$ "$X \le Y$" with probability one or almost surely

then $E(X) \le E(Y)$. $P(\text{"some property"}) = 1 \Leftrightarrow$ "some property"

$A = \{x \le y\}$



$P(E_A) = 1$
$P(E_A^c) = 0$

Proof. First, if Z is a random variable with finite mean and

$$P(Z \ge 0) = 1,$$ "$Z \ge 0$" with probability one or almost surely

then $E(Z) = \int_0^\infty z\, dF_Z(z) \ge 0$. $\quad F_Z(z) = 0$, when $z < 0$ ($z \ge 0$)

For the general case, let $Z = Y - X$, then $Z \ge 0$ with probability one, and therefore, $0 \le E(Z) = E(Y-X) = E(Y) - E(X)$.

➢ Corollary. If $P(a \le X \le b) = 1$ for some constants a, b, then

$P(X - a \ge 0) = 1 \to 0 \le E(X-a) = E(X) - a \to$ $$a \le E(X) \le b.$$ intuition

• Theorem. If two random vectors $\mathbf{X}$ $(\in \mathbb{R}^m)$ and $\mathbf{Y}$ $(\in \mathbb{R}^n)$ are independent (i.e., $F_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y}) = F_{\mathbf{X}}(\mathbf{x}) \times F_{\mathbf{Y}}(\mathbf{y})$, or

$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \times f_{\mathbf{Y}}(\mathbf{y}), \text{ or } p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y}) = p_{\mathbf{X}}(\mathbf{x}) \times p_{\mathbf{Y}}(\mathbf{y})\ ),$$

exchange of E & Σ

cf.

then for g: $\mathbb{R}^m \to \mathbb{R}$ and h: $\mathbb{R}^n \to \mathbb{R}$, $g \times h: \mathbb{R}^{m+n} \to \mathbb{R}$

Note. $g(\mathbf{X})$ & $h(\mathbf{Y})$ are independent. (LNp.7-24)

exchange of E & Π

$$E[g(\mathbf{X}) \times h(\mathbf{Y})] = E[g(\mathbf{X})] \times E[h(\mathbf{Y})].$$

Proof. We only prove it for the continuous case: similar proof for discrete, but $\int \to \Sigma$, $f \to p$

$$E[g(\mathbf{X})h(\mathbf{Y})] = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} g(\mathbf{x})h(\mathbf{y}) f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})\, d\mathbf{y}\, d\mathbf{x}$$

∵ independent

$$\overset{\because}{=} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} g(\mathbf{x})h(\mathbf{y}) f_{\mathbf{X}}(\mathbf{x}) f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}\, d\mathbf{x}$$

product of function of x, function of y

$$= \int_{\mathbb{R}^m} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) \left[\int_{\mathbb{R}^n} h(\mathbf{y}) f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}\right] d\mathbf{x}$$

This is a constant for x

$$= \left[\int_{\mathbb{R}^m} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}\right] \left[\int_{\mathbb{R}^n} h(\mathbf{y}) f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}\right]$$

$$= E[g(\mathbf{X})] E[h(\mathbf{Y})].$$

independent: $E(X_1 \times X_2 \times \cdots \times X_n) = E(X_1)E(X_2)\cdots E(X_n)$ generalization

independent $\Rightarrow$ uncorrelated ($\Leftarrow$ ✗)

➢ Corollary. For 2 independent r.v.'s X and Y,

exchange of E & Π

$$E(XY) = E(X) \times E(Y).$$ This is called "X and Y are uncorrelated"

Proof. Let $g(X) = X$ and $h(Y) = Y$. check 3. in LNp.8-8

➢ **Q**: For independent r.v.'s X and Y, $E(X/Y) \neq E(X)/E(Y)$?

linear transformation like a+bY ← cf. → h: concave function (convex?)

$E[h(Y)] < h(E(Y))$

$E\left(\frac{X}{Y}\right) = E\left(X \cdot \frac{1}{Y}\right)$ ∵ X & $\frac{1}{Y}$ are indep. (LNp.7-24)

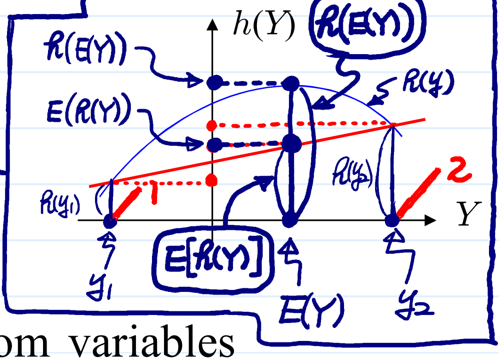
$= E(X) \cdot E\left(\frac{1}{Y}\right) \overset{?}{=} E(X) \cdot \frac{1}{E(Y)}$

➢ Note. $E[h(Y)] \neq h(E(Y))$ in general, e.g.,

Are h & E exchangable?

$E(1/Y) \neq 1/E(Y)$.

$h(y) = -\frac{1}{y}$ — convex, $y>0$; concave, $y<0$



⊙ Covariance and Correlation between 2 random variables → joint distribution

Recall: mean & variance (defined in terms of E)

➢ Definition. Suppose that X and Y are two random variables with finite means μ_X, μ_Y and variances σ_X^2, σ_Y^2, respectively.

They can be calculated from the marginal distributions of X & Y.

1. Let $g(x, y) = (x-\mu_X)(y-\mu_Y)$, then $g: \mathbb{R}^2 \to \mathbb{R}^1$

$$Cov(X,Y) \equiv E[g(X,Y)]$$

$$= E[(X-\mu_X)(Y-\mu_Y)]$$

Its calculation needs the joint dist.

fixed value, not random

is called the *covariance* between X and Y, denoted by σ_{XY}.

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2. The *correlation* (*coefficient*) between X and Y is defined as

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$$Cor(X,Y) = \sigma_{XY}/(\sigma_X \sigma_Y)$$

standard deviation (LNp.5-16, 6-14)

and denoted by ρ_{XY}. cov(X,Y)

3. X and Y are called *uncorrelated* if $\rho_{XY}=0$. $\xleftrightarrow{\text{iff}}$ cov(X,Y)=0

- A special case of covariance:

$Cov(X, X) \overset{\Downarrow}{=} Var(X)$. — $E[(X-\mu_X)(X-\mu_X)] = E[(X-\mu_X)^2]$

↳ Y=X

➢ Intuitive explanation of covariance and correlation

by definition $E[(X-\mu_X)(Y-\mu_Y)]$

▪ Covariance is the average value of the product of the deviation of X from its mean and the deviation of Y from its mean.

check the 1st graph in LNp.8-9

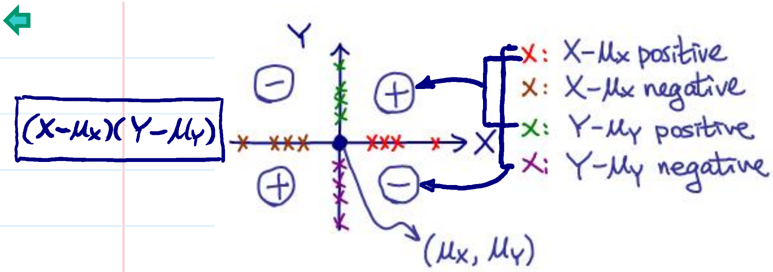
▪ Covariance is a measure of the *joint* variability of X and Y, or their degree of association.

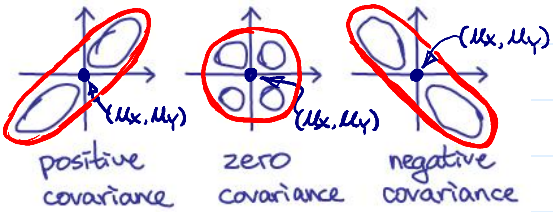
not necessarily a causal relationship

whether Y↑ (or Y↓) when X↑
e.g. X: height, Y: weight

▪ Positive Covariance and Negative Covariance

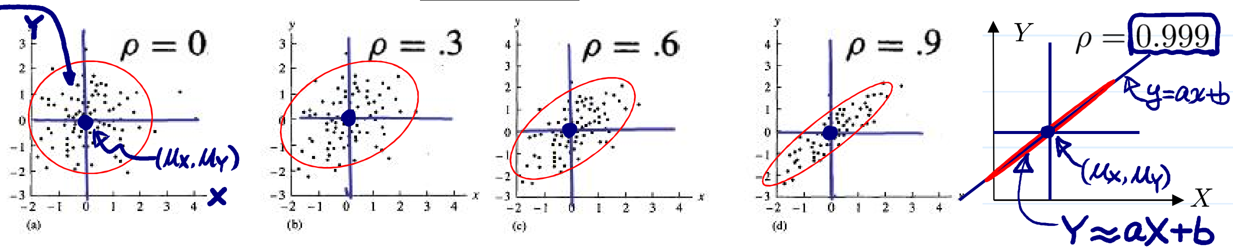
drawback: covariance depends on the unit/scale of X & Y
e.g. height: m → cm, |covariance| 10^2 times larger





- Correlation Coefficient is unit free. (why?) check its definition: $\frac{\sigma_{XY}}{\sigma_X \sigma_Y}$
- Correlation coefficient measures the strength of the *linear* relationship between X and Y.

the points reflect the joint distribution



Theorem. $Cov(X, Y) = E(XY) - \mu_X\mu_Y$. $\xleftrightarrow{\text{c.f.}}$ $cov(X,X) = Var(X) = E(X^2) - \mu_X^2$

Proof. $Cov(X, Y) = E[(X-\mu_X)(Y-\mu_Y)]$

cov = 0 ⇔ E(XY)=E(X)E(Y)

$$
\begin{aligned}
&= E(XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y) \\
&= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X\mu_Y \\
&= E(XY) - \mu_X\mu_Y - \cancel{\mu_Y\mu_X} + \cancel{\mu_X\mu_Y}.
\end{aligned}
$$

constants