➢ Example. If a sample of n balls is drawn without replacement from a box containing R red balls, W white balls, and $N-R-W$ blue balls. Let

generalization of hypergeometric ← without or with replacement (similar to the generalization from binomial to multinomial)

$n-X-Y$: # of blue balls

X = # of red balls in the sample,

Y = # of white balls in the sample,

then, the joint pmf of (X, Y) is

$$p_{X,Y}(x,y) = \frac{\binom{R}{x}\binom{W}{y}\binom{N-R-W}{n-x-y}}{\binom{N}{n}},$$

$x \in \{0, 1, \cdots, R\}$
$y \in \{0, 1, \cdots, W\}$
$0 \le x+y \le n$

Find $E_Y(Y)$.

Sol. Because $Y|X=x \sim$ hypergeometric$(n-x, N-R, W)$,

$g(x) \equiv E_{Y|X}(Y|X=x) = (n-x)[W/(N-R)]$. LNp.5-42

Because $X \sim$ hypergeometric$(n, N, R) \Rightarrow E_X(X) = n(R/N)$, and then

$$\begin{aligned} E_Y(Y) &= E_X[E_{Y|X}(Y|X)] = E_X[g(X)] \\ &= E_X\left[(n-X)\frac{W}{N-R}\right] = \frac{W}{N-R}[n - E_X(X)] \\ &= \frac{W}{N-R}\left(n - n\frac{R}{N}\right) = n\frac{W}{N}. \end{aligned}$$

$E_Y E_{X|Y} = E_{X,Y}$

Note that $Y \sim$ hypergeometric$(n, N, W) \Rightarrow E_Y(Y) = n(W/N)$.

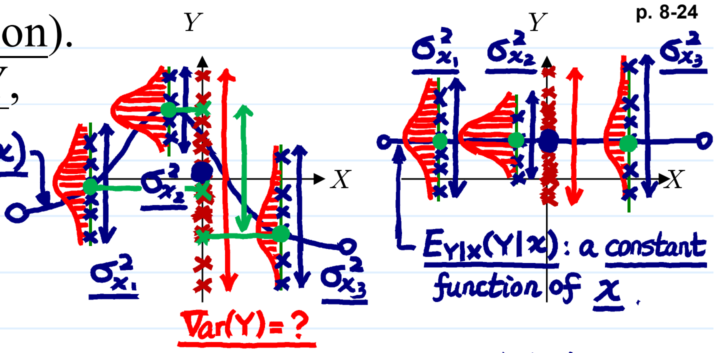
⦿ Theorem (Variance Decomposition). For two random vectors $\mathbf{X}$ and $\mathbf{Y}$,

$$\begin{aligned} Var_{\mathbf{Y}}(Y_i) &= Var_{Y_i}(Y_i) \\ &= Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})] \\ &\quad + E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]. \end{aligned}$$

"between group" variation

"within group" variation

$E_{Y|X}(Y|x)$



→ The concept behind the Thm leads to "Analysis of Variance (ANOVA)" in statistics.

Proof. $Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{x}) = E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{x}) - [E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{x})]^2$,

Note
$Var(Z)$
$= E(Z-\mu)^2$
$= E(Z^2) - [E(Z)]^2$

and, $E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$

$$= E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\}.$$

Also, $Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$ $\equiv g(X) = Z$ cf. same

$$= E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} - \{E_{\mathbf{X}}[\underbrace{E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})}_{g(X)}]\}^2.$$

Now, $Var_{\mathbf{Y}}(Y_i) = E_{\mathbf{Y}}(Y_i^2) - [E_{\mathbf{Y}}(Y_i)]^2$

Law of Total Expectation (LNp.8-22)

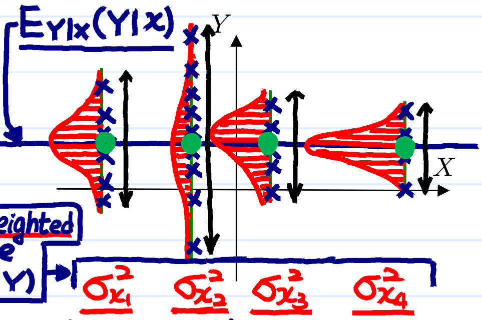
$$\begin{aligned} &\overset{E_{X,Y}}{=} E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - \{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]\}^2 \\ &= E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} \\ &\quad + E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} - \{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]\}^2 \\ &= E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})] + Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]. \end{aligned}$$

➢ Corollary.

- $Var_{\mathbf{Y}}(Y_i) \geq E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$ and the equality holds if and only if

$$E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X}) = E_{\mathbf{Y}}(Y_i)$$

with probability one.



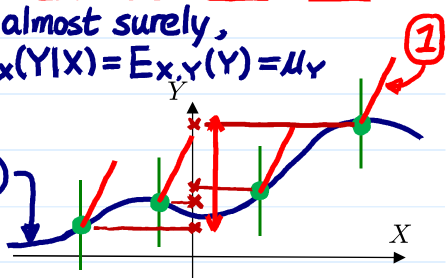
$Var_x[\underbrace{E_{Y|X}(Y|X)}_{g(X)}] = 0 \Rightarrow g(x) = E_{Y|X}(Y|x)$ is a constant over x almost surely, and the constant is $E_x[g(X)] = E_x E_{Y|X}(Y|X) = E_{X,Y}(Y) = \mu_Y$

$Y = g(X)$
$Var_Y(Y) = Var_X(g(X))$

- $Var_{\mathbf{Y}}(Y_i) \geq Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$ and the equality hold if and only if $g(x) = E_{Y|X}(Y|x)$

$g^*(X) = Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X}) = 0$ ($\Rightarrow Y_i = E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})$)

with probability one. $E_X[Var_{Y|X}(Y|X)] = 0$, $\quad g^*(X) \geq 0$



$\sigma^2_{x_1} = \sigma^2_{x_2} = \sigma^2_{x_3} = \sigma^2_{x_4}$
$= 0 \quad = 0 \quad = 0 \quad = 0$

❖ **Reading**: textbook, Sec 7.5

Conditional Expectation and Prediction

- Problem formulation: predicting the value of a r.v. Y on the basis of the observed value of a r.v. X ← fixed   random

➢ Data: X and Y (example?)

X	身高	雨量	地點
Y	體重	米產量	房價

➢ Statistical modeling: assigning (X, Y) a (known) joint distribution (cdf $F(x, y)$, pdf $f(x, y)$, or pmf $p(x, y)$) — If unknown, a statistical problem

➢ Objective: predicting Y by using a function of X, i.e.,

When $X = x$, use $g(x)$ to predict value of Y → $g(X)$ ← predictor ← e.g., for example in LNp.8-21, g transform year into cm

how the distribution can help?

➢ Predictor: considering the following three groups of g's

(i) $G_1 = \{g(x) : g(x) = c,$ where $c \in \mathbb{R}\}$ ← not use the information of X

(ii) $G_2 = \{g(x) : g(x) = a + bx,$ where $a, b \in \mathbb{R}\}$

(iii) $G_3 = \{g(x) : g$ is an arbitrary function$\}$

Note. $G_1 \subset G_2 \subset G_3$.

best in G_2 is no worse than best in G_1
best in G_3 : : : : best in G_2

➢ Question: Within each group, what is the "best" predictor?

how to choose c in G_1?
 : : : a, b in G_2?
 : : : g in G_3?

➢ Criterion: minimizing mean square error

meaning? → $\text{MSE} \equiv E_{X,Y}\{[Y - g(X)]^2\}$

Why taking expectation over X?   true value   error   predicted value

G1 **Theorem** (best *constant* predictor under MSE). a constant function of x minimum

$$E_{X,Y}(Y-c)^2 = E_Y(Y-c)^2 \geq E_Y[Y - E_Y(Y)]^2 = Var_Y(Y)$$

The equality holds if and only if $c = E_Y(Y)$. only need to know μ_Y

Proof. $R(Y)=(Y-c)^2$: a function of Y

$E_{X,Y}[R(Y)] = E_Y[R(Y)]$ LNp.8-22

$\| $

$E_Y E_{X|Y}[R(Y)|Y]$ $R(Y)$

Thm in LNp.5-19

$$E_Y(Y-c)^2 = Var_Y(Y) + (\mu_Y - c)^2 \geq Var_Y(Y)$$

cf. LNp.8-24

1/4

G3 **Theorem** (best predictor under MSE). LNp.8-24 minimum

$$E_{X,Y}[Y-g(X)]^2 \geq E_{X,Y}[Y - E_{Y|X}(Y|X)]^2 = E_X[Var_{Y|X}(Y|X)]$$

(*)

The equality holds if and only if $g(x) = E_{Y|X}(Y|x)$. a function of X only cf. (LNp.8-21)

Proof.
$$
\begin{aligned}
E_{X,Y}[Y - g(X)]^2 &= E_{X,Y}\{[Y - E_{Y|X}(Y|X)] + [E_{Y|X}(Y|X) - g(X)]\}^2 \\
&= E_{X,Y}[Y - E_{Y|X}(Y|X)]^2 + \underbrace{E_X}_{E_{X,Y}}[E_{Y|X}(Y|X) - g(X)]^2 \\
&\quad + 2 \cdot \underbrace{E_{X,Y}\{[Y - E_{Y|X}(Y|X)][E_{Y|X}(Y|X) - g(X)]\}}_{=0} \\
&= E_{X,Y}[Y - E_{Y|X}(Y|X)]^2 + E_X[E_{Y|X}(Y|X) - g(X)]^2 \\
&\geq E_{X,Y}[Y - E_{Y|X}(Y|X)]^2
\end{aligned}
$$

a function of X only

last "="

$=0$ iff $g(x) = E_{Y|X}(Y|X)$