Function of $X_1, \dots, X_n$

- Corollary. Suppose that $X_1, \dots, X_n$ are uncorrelated and have same mean μ and variance σ^2. Let

relax i.i.d condition

$E(\overline{X}_n)=\mu$

$$\underbrace{S^2}_{\text{a r.v.}} = \frac{\sum_{i=1}^n (X_i - \overline{X}_n)^2}{n-1}, \overset{\text{c.f.}}{\longleftrightarrow} \boxed{\text{definition of variance}} = E(X-\mu)^2 = \sigma^2$$

c.f.

then $E(S^2)=\sigma^2$. $\Rightarrow S^2 \xrightarrow{e} \sigma^2$

It can be shown that when $n\to\infty$ $Var(S^2)\to 0$

Note. $\overline{X}_n \xrightarrow{e} \mu$, $E(\overline{X}_n)=\mu$, $Var(\overline{X}_n)=\sigma^2/n$

Proof. $(n-1)S^2 = \sum_{i=1}^n (X_i \overbrace{-\ }^{-\mu+\mu} \overline{X}_n)^2$

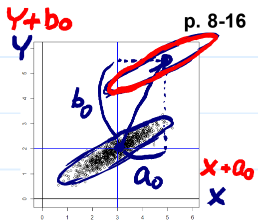
$$\begin{aligned}
&= \textstyle\sum_{i=1}^n [(X_i-\mu)-(\overline{X}_n-\mu)]^2 \\
&= [\textstyle\sum_{i=1}^n (X_i-\mu)^2] + [\textstyle\sum_{i=1}^n (\overline{X}_n-\mu)^2] \\
&\quad - 2(\overline{X}_n-\mu)\underbrace{[\textstyle\sum_{i=1}^n (X_i-\mu)]}_{=n(\overline{X}_n-\mu)} \\
&= [\textstyle\sum_{i=1}^n (X_i-\mu)^2] + n(\overline{X}_n-\mu)^2 - 2n(\overline{X}_n-\mu)^2 \\
&= [\textstyle\sum_{i=1}^n (X_i-\mu)^2] - n(\overline{X}_n-\mu)^2.
\end{aligned}$$

random

Therefore,

$$\begin{aligned}
(\cancel{n-1})E(S^2) &= \left\{\sum_{i=1}^n \underbrace{E\left[(X_i-\mu)^2\right]}_{Var(X_i)=\sigma^2}\right\} - n\underbrace{E\left[(\overline{X}_n-\mu)^2\right]}_{Var(\overline{X}_n)} \\
&= n\sigma^2 - n\underbrace{Var(\overline{X}_n)}_{=\sigma^2/n\ (\text{LNp.8-13})} = (\cancel{n-1})\sigma^2.
\end{aligned}$$

- Note. The previous three corollaries also hold if $X_1, \dots, X_n$ are "uncorrelated" is replaced by "independent." $\leftarrow$ $\because$ "independence" implies "uncorrelated"



Recall 2nd Corollary in LNp.8-10

Theorem (ρ of linear transformation).

$$Cor(\underbrace{a_0}_{\text{gone}}+a_1X,\ \underbrace{b_0}_{\text{gone}}+b_1Y)=\text{sign}(a_1b_1)\times Cor(X, Y),$$

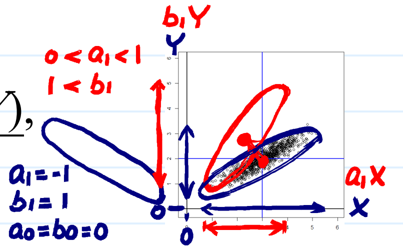
and

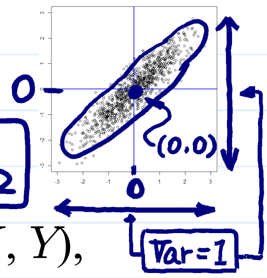
$$|Cor(a_0+a_1X,\ b_0+b_1Y)|=|Cor(X, Y)|,$$

Why?

i.e., $|\rho_{XY}|$ is invariant under location and scale changes.

standardization





$$\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right) = \left[\frac{(X+a_0)-(\mu_X+a_0)}{\sigma_X}\right] \times \left[\frac{(Y+b_0)-(\mu_Y+b_0)}{\sigma_Y}\right] = \left(\frac{a_1X-a_1\mu_X}{|a_1|\sigma_X}\right) \times \left(\frac{b_1Y-b_1\mu_Y}{|b_1|\sigma_Y}\right) \times \text{sign}(a_1b_1)$$

Proof. Let $S=a_0+a_1X$ and $T=b_0+b_1Y$, then

Thm in LNp.8-12

$$Cov(S, T)=Cov(a_0+a_1X,\ b_0+b_1Y)=a_1b_1Cov(X, Y),$$

$$Var(S)=a_1^2\,Var(X), \quad \text{and} \quad Var(T)=b_1^2\,Var(Y).$$

Therefore,

$$\rho_{ST} = \frac{Cov(S,T)}{\sigma_S\sigma_T} = \frac{a_1b_1Cov(X,Y)}{|a_1||b_1|\sigma_X\sigma_Y} = \boxed{\frac{a_1b_1}{|a_1b_1|}}\rho_{XY}.$$

$/\!/ \ \text{sign}(a_1b_1)$

$\frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

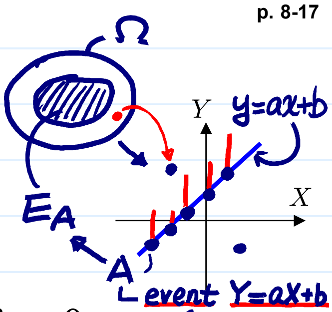
- Theorem (some properties of ρ). cf. → Cauchy-Schwarz inequality

$\boxed{0 \le |\rho_{XY}| \le 1}$ ⇔ (1) $-1 \le \rho_{XY} \le 1$. ($\Leftrightarrow |Cov(X,Y)| \le \sigma_X \sigma_Y$)

ρ is unit-free

(2) $\rho_{XY} = \pm 1$ if and only if there exist $a, b \in \mathbb{R}$ such that $P(Y=aX+b)=1$. $\boxed{Y=aX+b \text{ almost surely}}$

(3) Furthermore, $\rho_{XY}=1$, if $a>0$ and $\rho_{XY}=-1$, if $a<0$.



$P(E_A)=1$

$P(E_A^c)=0$

Proof of (1).

Cauchy-Schwarz inequality
$\underline{u}=(u_1,\cdots,u_n)$, $\underline{v}=(v_1,\cdots,v_n)$
$|\sum_i u_i v_i| = |\langle \underline{u}, \underline{v} \rangle| \le \|\underline{u}\| \cdot \|\underline{v}\| = \sqrt{\sum_i u_i^2}\sqrt{\sum_i v_i^2}$
$(Y(\omega)-\mu_Y)*\sqrt{p(\omega)}$
$(X(\omega)-\mu_X)*\sqrt{p(\omega)}$ ← $\sum_i \to \sum_{\omega \in \Omega}$

$$0 \le Var\left(\frac{X}{\sigma_X}+\frac{Y}{\sigma_Y}\right)$$

$\boxed{\text{Thm in LNp.8-13}}$

$$= Var\left(\frac{X}{\boxed{\sigma_X}}\right) + Var\left(\frac{Y}{\boxed{\sigma_Y}}\right) + 2Cov\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right)$$

constants

$$= \frac{Var(X)}{\sigma_X^2} + \frac{Var(Y)}{\sigma_Y^2} + 2\,\frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$= 1+1+2\rho_{XY} \quad \Rightarrow \rho_{XY} \ge -1.$$

Similarly,

$$0 \le Var\left(\frac{X}{\sigma_X}-\frac{Y}{\sigma_Y}\right) = 1+1-2\rho_{XY} \Rightarrow \rho_{XY} \le 1.$$

Proof of (2) and (3). We see from the proof of (1),

$$\rho_{XY}=1 \Leftrightarrow Var\left(\frac{X}{\sigma_X}-\frac{Y}{\sigma_Y}\right)=0,$$

$\boxed{Var(Z)=0 \Leftrightarrow Z=c \text{ almost surely, for a constant } c}$

$$\Leftrightarrow \quad P\left(\frac{X}{\sigma_X}-\frac{Y}{\sigma_Y}=c\right)=1,$$

where c is a constant.

$$\Leftrightarrow \quad P\left(Y=\boxed{\frac{\sigma_Y}{\sigma_X}}X+c\sigma_Y\right)=1. \quad >0$$

Similarly, $\rho_{XY}=-1 \Leftrightarrow P\left(Y=\boxed{-\frac{\sigma_Y}{\sigma_X}}X+c\sigma_Y\right)=1.$ <0

$\boxed{\text{dist. unknown}}$

- different transformation g ⇒ different information
- These expections are called parameters in statistics
- parameters can be estimated by r.v.'s (transformation of data), e.g., $\bar{X}_n \xrightarrow{e} \mu$, $S^2 \xrightarrow{e} \sigma^2$

- **Q**: How to use expectations to (roughly) characterize the distribution of random variables $X_1, \ldots, X_n$?

$\boxed{g\text{: 1st order polynomials of } X_1,\cdots,X_n}$

- $g(X_1, \ldots, X_n)=X_i \Rightarrow E[g(\mathbf{X})]=\mu_{X_i}$: *mean* of X_i.
- $g(X_1, \ldots, X_n)=(X_i-\mu_{X_i})^2 \Rightarrow E[g(\mathbf{X})]=\sigma_{X_i}^2$: *variance* of X_i.
- $g(X_1, \ldots, X_n)=(X_i-\mu_{X_i})(X_j-\mu_{X_j})$ for $i \ne j$
  $\Rightarrow E[g(\mathbf{X})]=\sigma_{X_iX_j}$: *covariance* of X_i and X_j.
- $g(X_1, \ldots, X_n)=[(X_i-\mu_{X_i})/\sigma_{X_i}][(X_j-\mu_{X_j})/\sigma_{X_j}]$ for $i \ne j$
  $\Rightarrow E[g(\mathbf{X})]=\rho_{X_iX_j}$: *correlation coefficient* of X_i and X_j.

$\boxed{g\text{: 2nd order polynomials of } X_1,\cdots,X_n}$

- Notes. $\mu_{X_i}, \sigma_{X_i}^2, \sigma_{X_iX_j}, \rho_{X_iX_j}$ are constants, not random

❖ **Reading**: textbook, Sec 7.1, 7.2, 7.4, 7.9

Recall. Conditional dist. LN p. 7-51 ~ 59

Conditional Expectation

- Recall. $p_{\cancel{\mathbf{Y}|\mathbf{X}}}(\mathbf{y}|\mathbf{x})$ or $f_{\cancel{\mathbf{Y}|\mathbf{X}}}(\mathbf{y}|\mathbf{x})$ is a pmf/pdf for $\mathbf{y}$ ($\mathbf{y}$: random, $\mathbf{x}$: fixed).
- Definition. For random vectors $\mathbf{X}$ and $\mathbf{Y}$, the *conditional expectation* of $Z=h(\mathbf{Y})$ given $\mathbf{X}=\mathbf{x}$, where h: $\mathbb{R}^m \rightarrow \mathbb{R}^1$, is

平均: $h(Y)$ 權重: $p_{Y|X}(y|x)$

$$E_{\mathbf{Y}|\mathbf{X}}\left(\underbrace{h(\mathbf{Y})}_{Z}\Big|\,\mathbf{X}=\mathbf{x}\right)=\sum_{\mathbf{y}\in\mathcal{Y}} h(\mathbf{y})\, p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}),$$

in the discrete case, or, $= \sum_z z\, p_{Z|X}(z|x)$

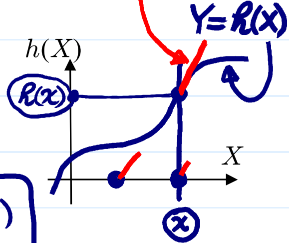
平均: $h(Y)$ 權重: $f_{Y|X}(y|x)$

$$E_{\mathbf{Y}|\mathbf{X}}\left(\underbrace{h(\mathbf{Y})}_{Z}\Big|\,\mathbf{X}=\mathbf{x}\right)=\int_{\mathbb{R}^m} h(\mathbf{y})\, f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y},$$

in the continuous case, $= \int_z z\, f_{Z|X}(z|x)\, dz$

provided that the sum or integral converges absolutely. cond. prob = 1

➢ Some Notes.



- $E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})\mid\mathbf{X}=\mathbf{x})$: a function of $\mathbf{x}$ and free of $\mathbf{Y}$.
- $E_{\mathbf{Y}|\mathbf{X}}[\,h(\mathbf{X})\mid\mathbf{X}=\mathbf{x}\,]=h(\mathbf{x})$.

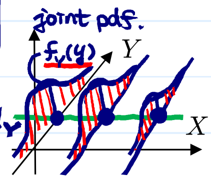
$E_{Y|X}[h(X,Y)|X=x]=\int h(x,y) f_{Y|X}(y|x)\,dy$

$\because f_{Y|X}(y|x)=f_Y(y)$, $p_{Y|X}(y|x)=p_Y(y)$

- If $\mathbf{X}$ and $\mathbf{Y}$ are independent, then

$$E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})|\mathbf{X}=\mathbf{x})=E_{\mathbf{Y}}[h(\mathbf{Y})].$$

a constant line of x → μ_Y



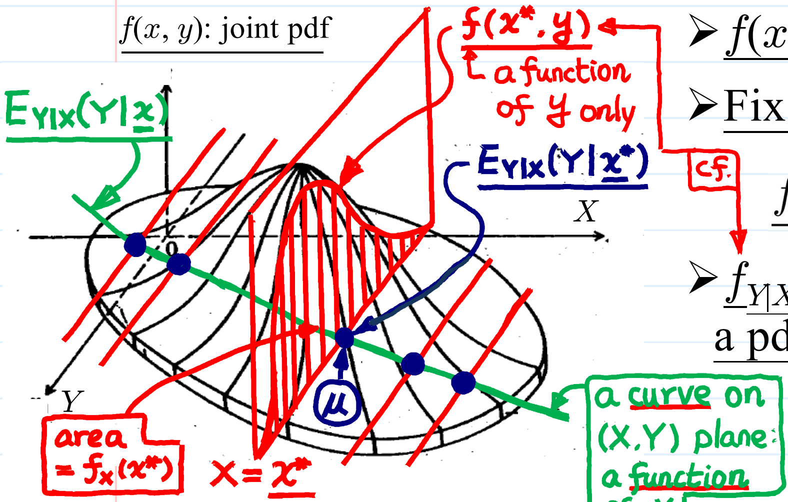
- Let $g(\mathbf{x})=E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}=\mathbf{x}]$, where g:$\mathbb{R}^n\rightarrow\mathbb{R}^1$, then we write

cf.

$$E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})|\mathbf{X})$$

cf. when $\mathbf{x}$ in g is replaced by $\mathbf{X}$ (a fixed value replaced by a r.v.).

- Notice that $g(\mathbf{X})$ is a random variable.



➢ $f(x, y)$: a joint pdf.

➢ Fix x^*, is $f(x^*, y)$ a pdf of y? i.e.,

cf.

$$f_X(x^*)=\int_{-\infty}^{\infty} f(x^*, y)\, dy \overset{?}{=} 1.$$

➢ $f_{Y|X}(y|x^*)=f(x^*, y)/f_X(x^*)$ is a pdf of y since

$$\frac{\int_{-\infty}^{\infty} \boxed{f(x^*, y)}\, dy}{\boxed{f_X(x^*)}}=1.$$

$h(Y)=Y$ center of gravity

➢ $E_{Y|X}(Y|x^*)$: mean of $f_{Y|X}(y|x^*)$.

➢ Do it for any $x=x^*$, and get a function of $x \Rightarrow E_{Y|X}(Y|x)$

$\int (y-\mu) f(x^*, y)\, dy = 0$

$\Rightarrow \int y f(x^*,y)\,dy = \mu \int f(x^*,y)\,dy = \mu f_X(x^*)$

$\Rightarrow \mu = \int y\, f(x^*,y)/f_X(x^*)\, dy = E_{Y|X}(Y|x^*)$

$\Rightarrow f_{Y|X}(y|x^*)$ & $f(x^*,y)$ have same center of gravity

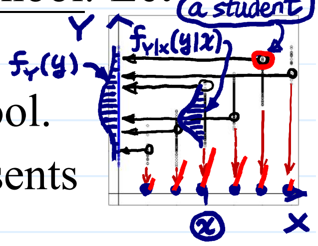
➢ Example. Sample a student from an elementary school. Let

X=age (unit: year), Y=height (unit: cm)

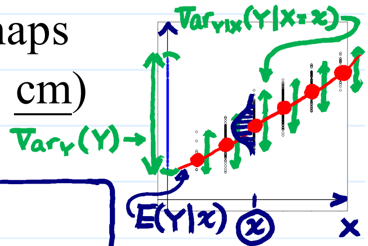
of the student. **Population**: all students of the school.

Q: What's the source of their randomness?

- $Y|X=x$: a random variable (unit: cm) that represents the height distribution of students with age=x.



- $g(x)=E_{Y|X}(Y|X=x)$ or $E_{Y|X}(Y|x)$: a *function* maps from age (unit: year) to average height (unit: cm) of students with age=x.

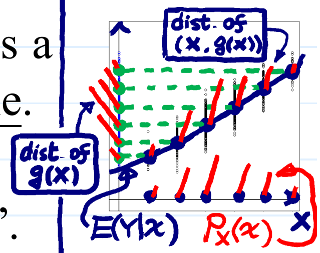


cf.

after cf. before

Note. $E_{Y|X}(Y|x)$ is not a random variable.

- $g(X)=E_{Y|X}(Y|X)$: a random variable because it is a function of age X, where X is a random variable.



$g(x)$: 1-to-1 in this case $\Rightarrow P(E_{Y|X}(Y|X)=E_{Y|X}(Y|x))=P(X=x)$

$R(Y)=[Y-g(x)]^2$

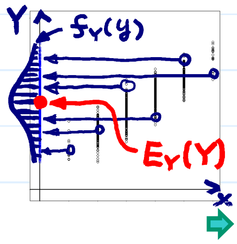
$E_{Y|X}[R(Y)|x]$

Note. $g(X)=E_{Y|X}(Y|X)$ is height, its unit is "cm".

- $Var_{Y|X}(Y|X=x)$ & $Var_{Y|X}(Y|X)$ defined similarly.

  - $E_Y(Y)$: average height of ***all*** students; cf.

    $Var_Y(Y)$: variation of height of ***all*** students. cf.



- Theorem (Law of Total Expectation). For two random vectors $\mathbf{X}$ ($\in\mathbb{R}^m$) and $\mathbf{Y}$ ($\in\mathbb{R}^n$),

$$E_{\mathbf{X},\mathbf{Y}}[R(\mathbf{Y})]=E_{\mathbf{Y}}E_{\mathbf{X}|\mathbf{Y}}$$

$$E_{\mathbf{X}}\{E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}]\}=E_{\mathbf{Y}}[h(\mathbf{Y})].$$

$E_{X|Y}[R(Y)]=R(Y)$

use the example in LNp.8-21 to realize the meaning of these terms.

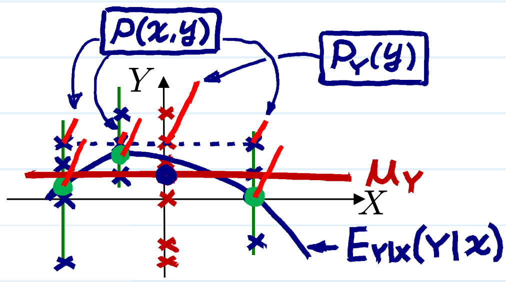
In particular, let $h(\mathbf{Y})=Y_i$, we have

$$\underbrace{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]}_{\equiv\, g(X)}=\underbrace{E_{\mathbf{Y}}(Y_i)}_{=\,E_{Y_i}(Y_i)=E_{X,Y}(Y_i)}.$$

Y & $E_{Y|X}(Y|X)$ have same mean

Proof.

(only prove it for the continuous case)



$$
\begin{aligned}
&E_{\mathbf{X}}\{E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}]\}\\
&= \int_{\mathbb{R}^m} E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})|\mathbf{x})\, f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}\\
&= \int_{\mathbb{R}^m}\left[\int_{\mathbb{R}^n} h(\mathbf{y}) f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y}\right] f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} h(\mathbf{y}) \frac{f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{\cancel{f_{\mathbf{X}}(\mathbf{x})}} \cancel{f_{\mathbf{X}}(\mathbf{x})}\, d\mathbf{x}d\mathbf{y}\\
&= \int_{\mathbb{R}^n} h(\mathbf{y})\left[\int_{\mathbb{R}^m} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})\, d\mathbf{x}\right] d\mathbf{y}\\
&= \int_{\mathbb{R}^n} h(\mathbf{y}) f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}\\
&= E_{\mathbf{Y}}[h(\mathbf{Y})].
\end{aligned}
$$

interchange dy & dx: $dy\,dx \to dx\,dy$

$E_{X,Y}[R(Y)]$

multiplication law

E_Y ① $\leftrightarrow$ dx ② $\updownarrow$ dy +

$E_XE_{Y|X}$ ① $\updownarrow$ dy + ② $\leftrightarrow$ dx +

generalization:

$E_{X,Y}[R(X,Y)]$

$=E_XE_{Y|X}[R(X,Y)|X]$

$=E_YE_{X|Y}[R(X,Y)|Y]$