Expectation

- Recall. Expectation for univariate random variable.
- Theorem. For random variables $\mathbf{X}=(X_1, \ldots, X_n)$ with joint pmf $p_{\mathbf{X}}$/pdf $f_{\mathbf{X}}$, the *expectation* of a univariate random variable Y, where

$$Y=g(X_1, \ldots, X_n), \qquad g:\mathbb{R}^n \to \mathbb{R}^1,$$

is

$$E(Y) \equiv \sum_{y\in\mathcal{Y}} y\, p_Y(y) \tag{1}$$

$$= \sum_{\mathbf{x}=(x_1,\ldots,x_n)\in\mathcal{X}} g(x_1,\ldots,x_n)\, p_{\mathbf{X}}(x_1,\ldots,x_n) \tag{2}$$

$$\equiv E[g(X_1,\ldots,X_n)]$$

if $X_1, \ldots, X_n$ are discrete and the sum converges absolutely, or

$$E(Y) \equiv \int_{-\infty}^{\infty} y f_Y(y)\, dy \tag{3}$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\ldots,x_n) f_{\mathbf{X}}(x_1,\ldots,x_n)\, dx_1\cdots dx_n \tag{4}$$

$$\equiv E[g(X_1,\ldots,X_n)]$$

if Y and $X_1, \ldots, X_n$ are continuous and the integrals converges absolutely.

Proof. Like the univariate case.

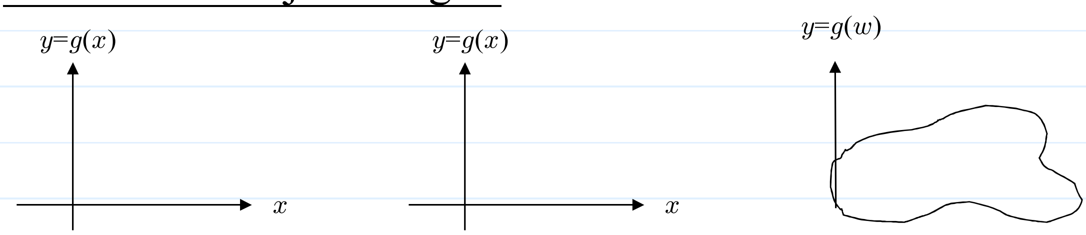
- **Q**: What if Y is discrete and $X_1, \ldots, X_n$ are continuous?
- Notation.
  - Shorthand notation. Combine (1) and (3) by writing

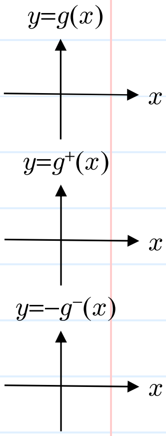
$$E(Y) = \int_{-\infty}^{\infty} y\, dF_Y(y) = \begin{cases} \sum_{y\in\mathcal{Y}} y\, p_Y(y), & \text{for discrete case,} \\ \int_{-\infty}^{\infty} y\, f_Y(y)\, dy, & \text{for continuous case,} \end{cases}$$

  and combine (2) and (4) by writing

$$E[g(\mathbf{X})] = \int_{\mathbb{R}^n} g(\mathbf{x})\, dF_{\mathbf{X}}(\mathbf{x}) = \begin{cases} \sum_{\mathbf{x}\in\mathcal{X}} g(\mathbf{x})\, p_{\mathbf{X}}(\mathbf{x}), & \text{for discrete case,} \\ \int_{\mathbb{R}^n} g(\mathbf{x})\, f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}, & \text{for continuous case.} \end{cases}$$

  - Riemann-Stieltjes Integral.





For example, for non-negative g, and non-decreasing, right-continuous F,

$$\int_a^b g(x)\, dF(x) = \lim \sum_{i=1}^{n} g(x_i)[F(x_i) - F(x_{i-1})].$$

where the limit is taken over all $a=x_0<x_1<\cdots<x_n=b$ as $n\to\infty$ and $\max_{i=1,\ldots,n}(x_i - x_{i-1}) \to 0$.

[Recall. The integral of g over (a, b] is defined as

$$\int_a^b g(x)\, dx = \lim \sum_{i=1}^{n} g(x_i)(x_i - x_{i-1}).]$$

➢ Note.

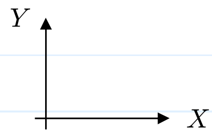
- $g(X_1, \ldots, X_n)=X_i \Rightarrow E[g(X_1, \ldots, X_n)]=E(X_i) \equiv \mu_{X_i}$.
- $g(X_1, \ldots, X_n)=(X_i-\mu_{X_i})^2 \Rightarrow E[g(X_1, \ldots, X_n)]=Var(X_i) \equiv \sigma^2_{X_i}$.

➢ Example (Average distance between two points). Suppose that X, Y are i.i.d. ~ Uniform(0, 1).

Let $D=|X-Y|$. Find $E(D)$.

- The joint pdf of (X, Y) is

$$f(x,y) = \begin{cases} 1, & 0 \le x \le 1, 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$



- $E(D) = \int_0^1 \int_0^1 |x-y|\, dydx = \int_0^1 \left[\int_0^x (x-y)\, dy + \int_x^1 (y-x)\, dy\right] dx$

$$= \int_0^1 \left[-\tfrac{1}{2}(y-x)^2\Big|_{y=0}^{x} + \tfrac{1}{2}(y-x)^2\Big|_{y=x}^{1}\right] dx$$

$$= \int_0^1 \tfrac{1}{2}\left[x^2 + (1-x)^2\right] dx = \tfrac{1}{6}\left[x^3 - (1-x)^3\right]\Big|_{x=0}^{1} = \tfrac{1}{3}.$$

• Theorem (Mean of Sum). For jointly distributed r.v.'s $X_1, \ldots, X_n$ and constants $-\infty < a_0, a_1, \ldots, a_n < \infty$,

$$E(a_0+a_1X_1+\cdots+a_nX_n) = a_0+a_1 E(X_1)+\cdots+a_n E(X_n).$$

Proof. $E(a_0 + a_1X_1 + \cdots + a_nX_n)$

$$\begin{aligned} &= \int_{\mathbb{R}^n} (a_0 + a_1x_1 + \cdots + a_nx_n)\, dF_{\mathbf{X}}(\mathbf{x}) \\ &= \int_{\mathbb{R}^n} a_0\, dF_{\mathbf{X}}(\mathbf{x}) + a_1 \int_{\mathbb{R}^n} x_1\, dF_{\mathbf{X}}(\mathbf{x}) \\ &\quad + \cdots + a_n \int_{\mathbb{R}^n} x_n\, dF_{\mathbf{X}}(\mathbf{x}) \\ &= a_0 + a_1E(X_1) + \cdots + a_nE(X_n). \end{aligned}$$

➢ Corollary. Suppose that $\mu=E(X_1)=\cdots=E(X_n)$. Let

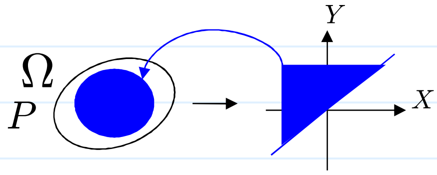
$$\overline{X}_n = \frac{X_1+\cdots+X_n}{n},$$

then, $E(\overline{X}_n) = \mu$.

➢ Corollary. If X and Y are r.v.'s with finite means and

$P(X \le Y)=1$,

then $E(X) \le E(Y)$.



Proof. First, if Z is a random variable with finite mean and

$P(Z \ge 0)=1$,

then $E(Z) = \int_0^\infty z\, dF_Z(z) \ge 0$.

For the general case, let $Z=Y-X$, then $Z \ge 0$ with probability one, and therefore, $0 \le E(Z) = E(Y-X) = E(Y)-E(X)$.

➢ Corollary. If $P(a \le X \le b)=1$ for some constants a, b, then

$$a \le E(X) \le b.$$

- Theorem. If two random vectors $\mathbf{X}$ ($\in\mathbb{R}^m$) and $\mathbf{Y}$ ($\in\mathbb{R}^n$) are independent (i.e., $F_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y}) = F_{\mathbf{X}}(\mathbf{x})\times F_{\mathbf{Y}}(\mathbf{y})$, or

  $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})=f_{\mathbf{X}}(\mathbf{x})\times f_{\mathbf{Y}}(\mathbf{y})$, or $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})=p_{\mathbf{X}}(\mathbf{x})\times p_{\mathbf{Y}}(\mathbf{y})$),

  then for g: $\mathbb{R}^m\rightarrow\mathbb{R}$ and h: $\mathbb{R}^n\rightarrow\mathbb{R}$,

  $$E[g(\mathbf{X})\times h(\mathbf{Y})] = E[g(\mathbf{X})]\times E[h(\mathbf{Y})].$$

Proof. We only prove it for the continuous case:

$$\begin{aligned}
E[g(\mathbf{X})h(\mathbf{Y})] &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} g(\mathbf{x})h(\mathbf{y}) f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \\
&= \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} g(\mathbf{x})h(\mathbf{y}) f_{\mathbf{X}}(\mathbf{x}) f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \\
&= \int_{\mathbb{R}^m} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) \left[\int_{\mathbb{R}^n} h(\mathbf{y}) f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}\right] d\mathbf{x} \\
&= \left[\int_{\mathbb{R}^m} g(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}\right] \left[\int_{\mathbb{R}^n} h(\mathbf{y}) f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}\right] \\
&= E[g(\mathbf{X})]E[h(\mathbf{Y})].
\end{aligned}$$

➢ Corollary. For 2 independent r.v.'s X and Y,

$$E(XY)=E(X)\times E(Y).$$

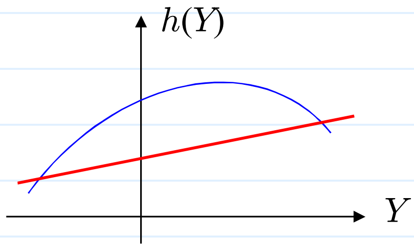
Proof. Let $g(X)=X$ and $h(Y)=Y$.

- **Q**: For independent r.v.'s X and Y,

  $E(X/Y)=E(X)/E(Y)$?

- Note. $E[h(Y)]\neq h(E(Y))$ in general, e.g.,

  $E(1/Y) \neq 1/E(Y)$.



• Covariance and Correlation between 2 random variables

- Definition. Suppose that X and Y are two random variables with finite means μ_X, μ_Y and variances σ_X^2, σ_Y^2, respectively.

  1.Let $g(x, y)=(x-\mu_X)(y-\mu_Y)$, then

$$\begin{aligned} Cov(X,Y) &\equiv E[g(X,Y)] \\ &= E[(X-\mu_X)(Y-\mu_Y)] \end{aligned}$$

  is called the *covariance* between X and Y, denoted by σ_{XY}.

  2.The *correlation* (*coefficient*) between X and Y is defined as

$$Cor(X,Y) = \sigma_{XY}/(\sigma_X\sigma_Y)$$

  and denoted by ρ_{XY}.

  3. X and Y are called *uncorrelated* if $\rho_{XY}=0$.

  ▪ A special case of covariance:

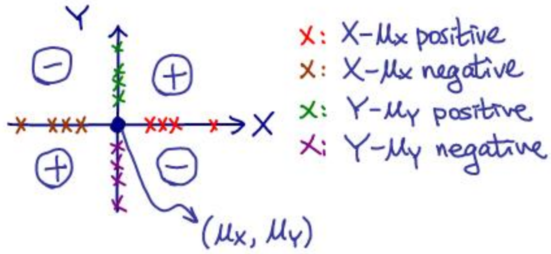
  $Cov(X, X) = Var(X)$.

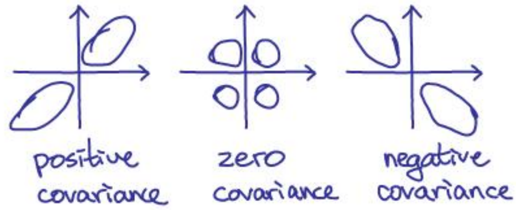
- Intuitive explanation of covariance and correlation

  ▪ Covariance is the average value of the product of the deviation of X from its mean and the deviation of Y from its mean.

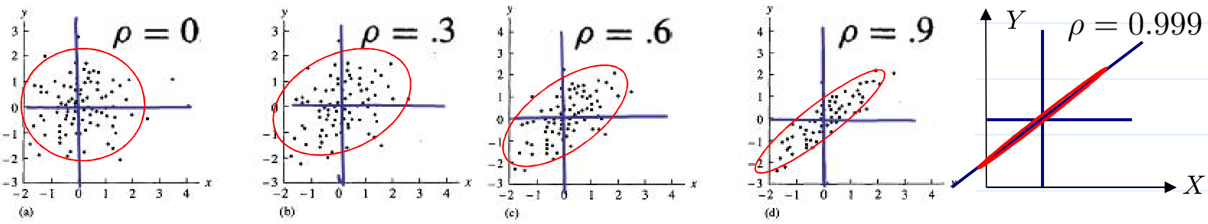
  ▪ Covariance is a measure of the *joint* variability of X and Y, or their degree of association.

  ▪ Positive Covariance and Negative Covariance





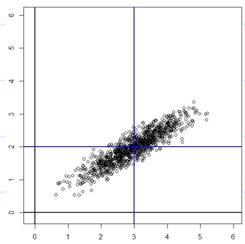
- Correlation Coefficient is unit free. (why?)
- Correlation coefficient measures the strength of the *linear* relationship between X and Y.



➢ Theorem. $Cov(X, Y) = E(XY) - \mu_X\mu_Y$.

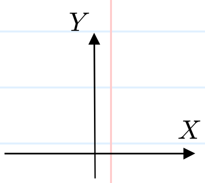
Proof.
$$\begin{aligned} Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E(XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y) \\ &= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X\mu_Y \\ &= E(XY) - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y. \end{aligned}$$

- Corollary. If X and Y are independent, then $Cov(X, Y)=0$, i.e., X and Y are uncorrelated.

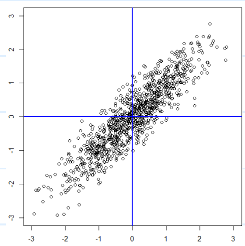


Proof. When X, Y are independent,

$$E(XY)=E(X)E(Y)=\mu_X\mu_Y.$$



- However, the converse statement is *not* necessarily true.



(e.g., let $X\sim$Uniform$(-1, 1)$ and $Y=X^2$, then

$Cov(X, Y)=0$,

but X and Y are *not* independent).

- Corollary.

$$\rho_{XY} = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right].$$

Proof. By definition.

➢ Example. If ($X_1, \ldots, X_m$) ~ Multinomial(n, m, $p_1, \ldots, p_m$), then

$$Cov(X_i, X_j) = -np_ip_j, \quad \text{for } 1 \le i \ne j \le m.$$

- Because (X_1, X_2, $X_3+\cdots+X_m$) ~ Multinomial(n, 3, p_1, p_2, $p_3+\cdots+p_m$), and

$$X_3+\cdots+X_m = n-X_1-X_2, \qquad p_3+\cdots+p_m = 1-p_1-p_2,$$

we have

$$\begin{aligned} E(X_1X_2) &= \sum x_1x_2\binom{n}{x_1,x_2,n-x_1-x_2}p_1^{x_1}p_2^{x_2}(1-p_1-p_2)^{n-x_1-x_2} \\ &= \sum x_1x_2\tfrac{n!}{x_1!x_2!(n-x_1-x_2)!}p_1^{x_1}p_2^{x_2}(1-p_1-p_2)^{n-x_1-x_2} \\ &= n(n-1)p_1p_2\left[\sum \tfrac{(n-2)!}{(x_1-1)!(x_2-1)!(n-x_1-x_2)!} \times (p_1)^{x_1-1}(p_2)^{x_2-1}(1-p_1-p_2)^{n-x_1-x_2}\right] \\ &= n(n-1)p_1p_2. \end{aligned}$$

- WLOG, we can get $E(X_iX_j) = n(n-1)p_ip_j$, for $i \ne j$.

Therefore, $$\begin{aligned} Cov(X_i, X_j) &= E(X_iX_j) - E(X_i)E(X_j) \\ &= n(n-1)p_ip_j - (np_i)(np_j) = -np_ip_j. \end{aligned}$$

- And, for $i \ne j$,

$$Cor(X_i, X_j) = \frac{-np_ip_j}{\sqrt{np_i(1-p_i)}\sqrt{np_j(1-p_j)}} = -\sqrt{\frac{p_ip_j}{(1-p_i)(1-p_j)}}.$$

- Cov & Cor for Sums of Random Variables

➢ Notation. In the following, let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be r.v.'s and $-\infty < a_0, a_1, \ldots, a_n$, $b_0, b_1, \ldots, b_m < \infty$ are constants.

➢ Recall. $E(a_0+a_1X_1+\cdots+a_nX_n) = a_0+a_1E(X_1)+\cdots+a_nE(X_n)$.

➢ Theorem (covariance of two sums).

$$Cov(a_0 + a_1X_1 + \cdots + a_nX_n, b_0 + b_1Y_1 + \cdots + b_mY_m) = \sum_{i=1}^n \sum_{j=1}^m a_ib_jCov(X_i, Y_j).$$

Proof. Let $S = a_0+a_1X_1+\cdots+a_nX_n$, and $T = b_0+b_1Y_1+\cdots+b_mY_m$, then

$$\begin{aligned} S - E(S) &= \sum_{i=1}^n a_i(X_i - \mu_{X_i}), \\ T - E(T) &= \sum_{j=1}^m b_j(Y_j - \mu_{Y_j}), \\ [S - E(S)][T - E(T)] &= \sum_{i=1}^n \sum_{j=1}^m a_ib_j(X_i - \mu_{X_i})(Y_j - \mu_{Y_j}). \end{aligned}$$

Therefore, $Cov(S,T) = E\{[S - E(S)][T - E(T)]\}$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j E[(X_i - \mu_{X_i})(Y_j - \mu_{Y_j})]$$

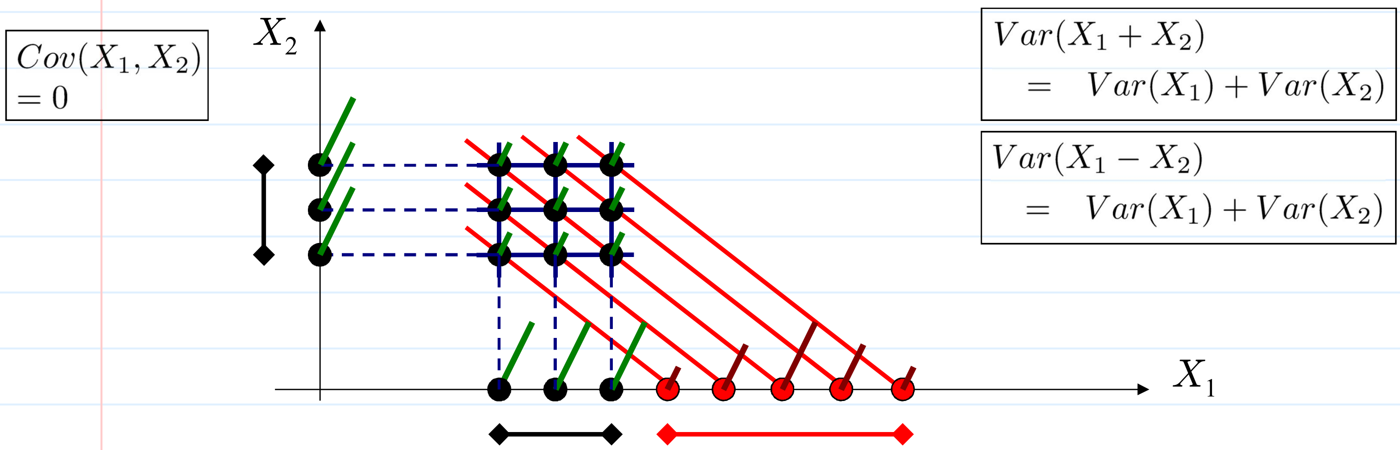
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j Cov(X_i, Y_j).$$

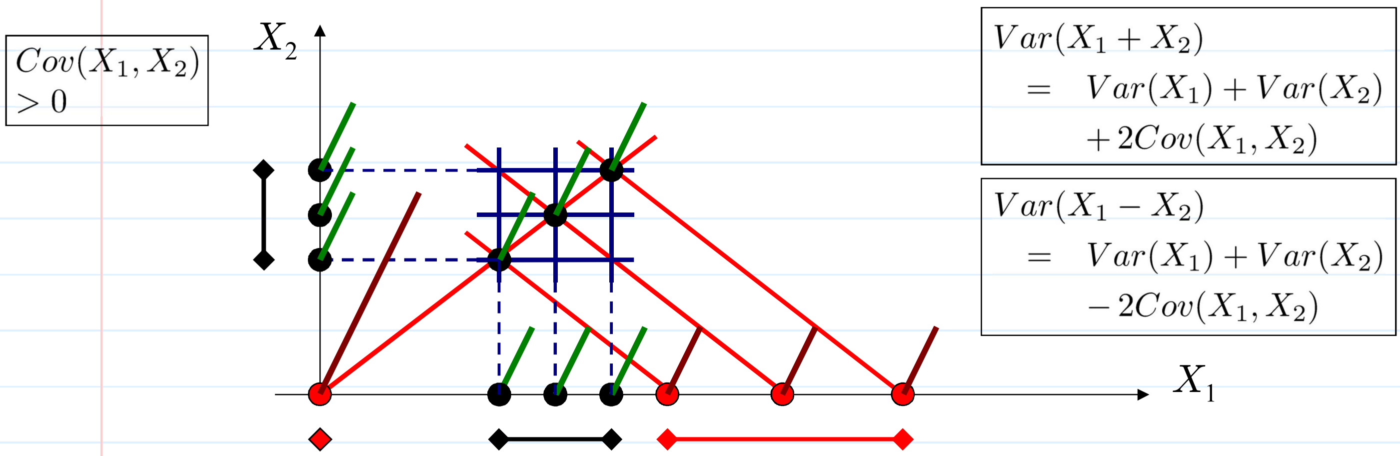
➢ Theorem (variance of sum).

$$Var(a_0 + a_1 X_1 + \cdots + a_n X_n)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j Cov(X_i, X_j)$$
$$= \sum_{i=1}^{n} a_i^2 Var(X_i) + 2 \sum_{1 \le i < j \le n} a_i a_j Cov(X_i, X_j).$$

Proof. $Cov(X_i, X_i) = Var(X_i)$ and $Cov(X_i, X_j) = Cov(X_j, X_i)$.

- Corollary. If $X_1, \ldots, X_n$ are uncorrelated, then
  $$Var(a_0 + a_1 X_1 + \cdots + a_n X_n) = \sum_{i=1}^{n} a_i^2 Var(X_i).$$
- Corollary. If $X_1, \ldots, X_n$ are uncorrelated and
  $$Var(X_1) = \cdots = Var(X_n) \equiv \sigma^2 < \infty,$$
  then $Var(\overline{X}_n) = \sigma^2/n$.





- Corollary. Suppose that $X_1, \ldots, X_n$ are uncorrelated and have same mean μ and variance σ^2. Let

$$S^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}{n-1},$$

then $E(S^2)=\sigma^2$.

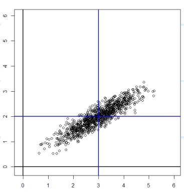
Proof. $(n-1)S^2 = \sum_{i=1}^{n}(X_i - \overline{X}_n)^2$

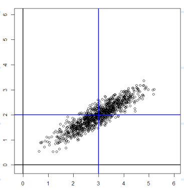
$$\begin{aligned}
&= \sum_{i=1}^{n}[(X_i-\mu)-(\overline{X}_n-\mu)]^2 \\
&= \left[\sum_{i=1}^{n}(X_i-\mu)^2\right] + \left[\sum_{i=1}^{n}(\overline{X}_n-\mu)^2\right] \\
&\quad - 2(\overline{X}_n-\mu)\left[\sum_{i=1}^{n}(X_i-\mu)\right] \\
&= \left[\sum_{i=1}^{n}(X_i-\mu)^2\right] + n(\overline{X}_n-\mu)^2 - 2n(\overline{X}_n-\mu)^2 \\
&= \left[\sum_{i=1}^{n}(X_i-\mu)^2\right] - n(\overline{X}_n-\mu)^2.
\end{aligned}$$

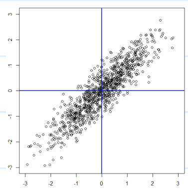
Therefore,

$$\begin{aligned}
(n-1)E(S^2) &= \left\{\sum_{i=1}^{n} E\left[(X_i-\mu)^2\right]\right\} - nE\left[(\overline{X}_n-\mu)^2\right] \\
&= n\sigma^2 - nVar(\overline{X}_n) = (n-1)\sigma^2.
\end{aligned}$$

- Note. The previous three corollaries also hold if $X_1, \ldots, X_n$ are "uncorrelated" is replaced by "independent."







➢ Theorem (ρ of linear transformation).

$Cor(a_0+a_1X,\ b_0+b_1Y)=\text{sign}(a_1b_1)\times Cor(X, Y)$,

and

$|Cor(a_0+a_1X,\ b_0+b_1Y)|=|Cor(X, Y)|$,

i.e., $|\rho_{XY}|$ is invariant under location and scale changes.

Proof. Let $S=a_0+a_1X$ and $T=b_0+b_1Y$, then

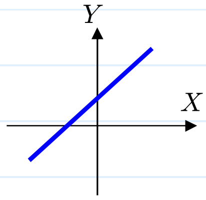
$Cov(S, T)=Cov(a_0+a_1X,\ b_0+b_1Y)=a_1b_1Cov(X, Y)$,

$Var(S)= a_1^2\, Var(X)$, and $Var(T)= b_1^2\, Var(Y)$.

Therefore,

$$\rho_{ST} = \frac{Cov(S,T)}{\sigma_S\sigma_T} = \frac{a_1b_1Cov(X,Y)}{|a_1||b_1|\sigma_X\sigma_Y} = \frac{a_1b_1}{|a_1b_1|}\rho_{XY}.$$

- Theorem (some properties of ρ).
  (1) $-1 \leq \rho_{XY} \leq 1$. ($\Leftrightarrow |Cov(X,Y)| \leq \sigma_X \sigma_Y$)
  (2) $\rho_{XY} = \pm 1$ if and only if there exist $a, b \in \mathbb{R}$ such that $P(Y=aX+b)=1$.
  (3) Furthermore, $\rho_{XY}=1$, if $a>0$ and $\rho_{XY}=-1$, if $a<0$.



Proof of (1).

$$
\begin{aligned}
0 \leq Var\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right)
&= Var\left(\frac{X}{\sigma_X}\right) + Var\left(\frac{Y}{\sigma_Y}\right) + 2Cov\left(\frac{X}{\sigma_X}, \frac{Y}{\sigma_Y}\right) \\
&= \frac{Var(X)}{\sigma_X^2} + \frac{Var(Y)}{\sigma_Y^2} + 2\,\frac{Cov(X,Y)}{\sigma_X \sigma_Y} \\
&= 1 + 1 + 2\,\rho_{XY} \quad \Rightarrow \rho_{XY} \geq -1.
\end{aligned}
$$

Similarly,

$$0 \leq Var\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = 1 + 1 - 2\rho_{XY} \Rightarrow \rho_{XY} \leq 1.$$

Proof of (2) and (3). We see from the proof of (1),

$$
\begin{aligned}
\rho_{XY} = 1 \Leftrightarrow Var\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) &= 0, \\
\Leftrightarrow \quad P\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y} = c\right) &= 1,
\end{aligned}
$$

where c is a constant.

$$\Leftrightarrow \quad P\left(Y = \frac{\sigma_Y}{\sigma_X} X + c\sigma_Y\right) = 1.$$

Similarly, $\rho_{XY} = -1 \quad \Leftrightarrow \quad P\left(Y = -\frac{\sigma_Y}{\sigma_X} X + c\sigma_Y\right) = 1.$

- **Q**: How to use expectations to (roughly) characterize the distribution of random variables $X_1, \ldots, X_n$?
  - $g(X_1, \ldots, X_n) = X_i \Rightarrow E[g(\mathbf{X})] = \mu_{X_i}$: *mean* of X_i.
  - $g(X_1, \ldots, X_n) = (X_i - \mu_{X_i})^2 \Rightarrow E[g(\mathbf{X})] = \sigma^2_{X_i}$: *variance* of X_i.
  - $g(X_1, \ldots, X_n) = (X_i - \mu_{X_i})(X_j - \mu_{X_j})$ for $i \neq j$
    $\Rightarrow E[g(\mathbf{X})] = \sigma_{X_i X_j}$: *covariance* of X_i and X_j.
  - $g(X_1, \ldots, X_n) = [(X_i - \mu_{X_i})/\sigma_{X_i}][(X_j - \mu_{X_j})/\sigma_{X_j}]$ for $i \neq j$
    $\Rightarrow E[g(\mathbf{X})] = \rho_{X_i X_j}$: *correlation coefficient* of X_i and X_j.
  - Notes. $\mu_{X_i}, \sigma^2_{X_i}, \sigma_{X_i X_j}, \rho_{X_i X_j}$ are constants, not random

❖ **Reading**: textbook, Sec 7.1, 7.2, 7.4, 7.9

Conditional Expectation

- Recall. $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ or $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ is a pmf/pdf for $\mathbf{y}$ ($\mathbf{y}$: random, $\mathbf{x}$: fixed).
- Definition. For random vectors $\mathbf{X}$ and $\mathbf{Y}$, the *conditional expectation* of Z=$h(\mathbf{Y})$ given $\mathbf{X}$=$\mathbf{x}$, where h: $\mathbb{R}^m \rightarrow \mathbb{R}^1$, is

$$E_{\mathbf{Y}|\mathbf{X}}\left(h(\mathbf{Y}) \middle| \mathbf{X}=\mathbf{x}\right) = \sum_{\mathbf{y}\in\mathcal{Y}} h(\mathbf{y})\, p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}),$$

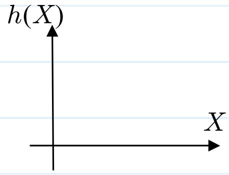
in the discrete case, or,

$$E_{\mathbf{Y}|\mathbf{X}}\left(h(\mathbf{Y}) \middle| \mathbf{X}=\mathbf{x}\right) = \int_{\mathbb{R}^m} h(\mathbf{y})\, f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y},$$

in the continuous case,

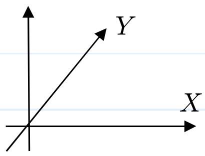
provided that the sum or integral converges absolutely.

- Some Notes.
  - $E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y}) \mid \mathbf{X}=\mathbf{x})$: a function of $\mathbf{x}$ and free of $\mathbf{Y}$.
  - $E_{\mathbf{Y}|\mathbf{X}}[\, h(\mathbf{X}) \mid \mathbf{X}=\mathbf{x}\,]=h(\mathbf{x})$.



  - If $\mathbf{X}$ and $\mathbf{Y}$ are independent, then

$$E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})|\mathbf{X}=\mathbf{x})=E_{\mathbf{Y}}[h(\mathbf{Y})].$$

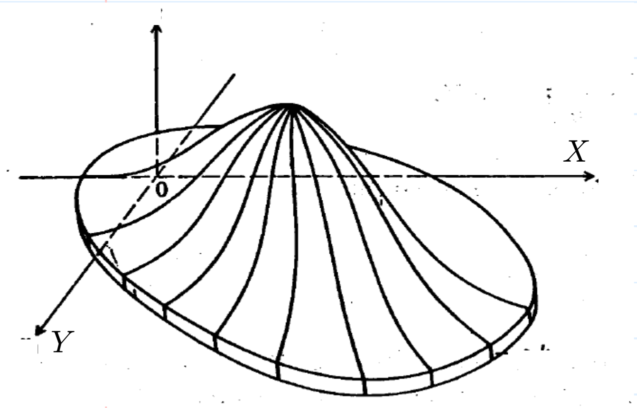


- Let $g(\mathbf{x})$=$E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}=\mathbf{x}]$, where g:$\mathbb{R}^n \rightarrow \mathbb{R}^1$, then we write

$$E_{\mathbf{Y}|\mathbf{X}}(h(\mathbf{Y})|\mathbf{X})$$

when $\mathbf{x}$ in g is replaced by $\mathbf{X}$ (a fixed value replaced by a r.v.).
  - Notice that $g(\mathbf{X})$ is a random variable.

$f(x, y)$: joint pdf



- $f(x, y)$: a joint pdf.
- Fix x^*, is $f(x^*, y)$ a pdf of y? i.e.,

$$f_X(x^*) = \int_{-\infty}^{\infty} f(x^*, y)\, dy \overset{?}{=} 1.$$

- $f_{Y|X}(y|x^*)$=$f(x^*, y)/f_X(x^*)$ is a pdf of y since

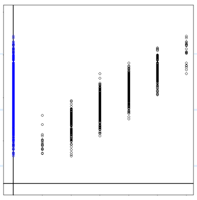
$$\frac{\int_{-\infty}^{\infty} f(x^*, y)\, dy}{f_X(x^*)} = 1.$$

- $E_{Y|X}(Y\,|x^*)$: mean of $f_{Y|X}(y|x^*)$.
- Do it for any x=x^*, and get a function of $x \Rightarrow E_{Y|X}(Y\,|x)$

➢ Example. Sample a student from an elementary school. Let

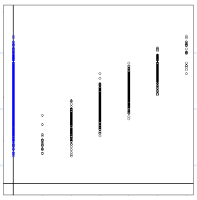
X=age (unit: year), Y=height (unit: cm)

of the student. **Population**: all students of the school.



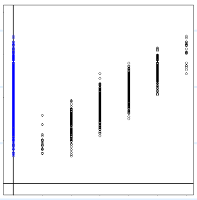
- $Y|X=x$: a random variable (unit: cm) that represents the height distribution of students with age=x.
- $g(x)=E_{Y|X}(Y|X=x)$ or $E_{Y|X}(Y|x)$: a *function* maps from age (unit: year) to average height (unit: cm) of students with age=x.

  Note. $E_{Y|X}(Y|x)$ is not a random variable.



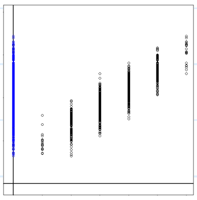
- $g(X)=E_{Y|X}(Y|X)$: a random variable because it is a function of age X, where X is a random variable.

  Note. $g(X)=E_{Y|X}(Y|X)$ is height, its unit is "cm".



- $Var_{Y|X}(Y|X=x)$ & $Var_{Y|X}(Y|X)$ defined similarly.
- $E_Y(Y)$: average height of ***all*** students;

  $Var_Y(Y)$: variation of height of ***all*** students.



- Theorem (Law of Total Expectation). For two random vectors $\mathbf{X}$ ($\in\mathbb{R}^m$) and $\mathbf{Y}$ ($\in\mathbb{R}^n$),

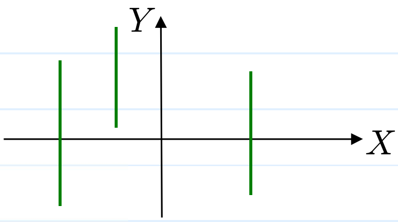
$$E_{\mathbf{X}}\{E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}]\}=E_{\mathbf{Y}}[h(\mathbf{Y})].$$

In particular, let $h(\mathbf{Y})=Y_i$, we have

$$E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]=E_{\mathbf{Y}}(Y_i).$$

Proof.

(only prove it for the continuous case)



$$\begin{aligned}
&E_{\mathbf{X}}\left\{E_{\mathbf{Y}|\mathbf{X}}[h(\mathbf{Y})|\mathbf{X}]\right\}\\
&= \int_{\mathbb{R}^m} E_{\mathbf{Y}|\mathbf{X}}\left(h(\mathbf{Y})|\mathbf{x}\right) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}\\
&= \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^n} h(\mathbf{y}) f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y}\right] f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} h(\mathbf{y}) \frac{f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{f_{\mathbf{X}}(\mathbf{x})} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}d\mathbf{y}\\
&= \int_{\mathbb{R}^n} h(\mathbf{y}) \left[\int_{\mathbb{R}^m} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})\, d\mathbf{x}\right] d\mathbf{y}\\
&= \int_{\mathbb{R}^n} h(\mathbf{y}) f_{\mathbf{Y}}(\mathbf{y})\, d\mathbf{y}\\
&= E_{\mathbf{Y}}\left[h(\mathbf{Y})\right].
\end{aligned}$$

➢ Example. If a sample of n balls is drawn without replacement from a box containing R red balls, W white balls, and $N-R-W$ blue balls. Let

X = # of red balls in the sample,

Y = # of white balls in the sample,

then, the joint pmf of (X, Y) is

$$p_{X,Y}(x,y) = \frac{\binom{R}{x}\binom{W}{y}\binom{N-R-W}{n-x-y}}{\binom{N}{n}},$$

Find $E_Y(Y)$.

Sol. Because $Y|X=x \sim$ hypergeometric$(n-x, N-R, W)$,

$$g(x) \equiv E_{Y|X}(Y|X=x) = (n-x)[W/(N-R)].$$

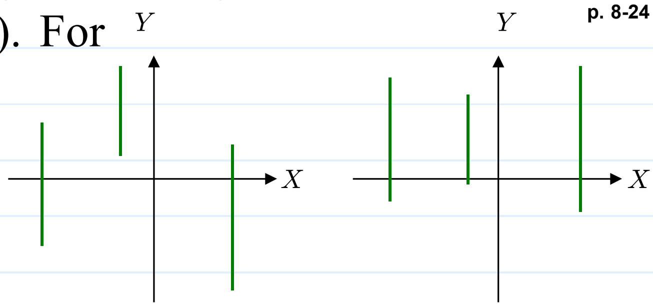
Because $X \sim$ hypergeometric$(n, N, R) \Rightarrow E_X(X)=n(R/N)$, and then

$$\begin{aligned} E_Y(Y) &= E_X[E_{Y|X}(Y|X)] = E_X[g(X)] \\ &= E_X\left[(n-X)\frac{W}{N-R}\right] = \frac{W}{N-R}[n - E_X(X)] \\ &= \frac{W}{N-R}\left(n - n\frac{R}{N}\right) = n\frac{W}{N}. \end{aligned}$$

Note that $Y \sim$ hypergeometric$(n, N, W) \Rightarrow E_Y(Y)=n(W/N)$.

• Theorem (Variance Decomposition). For two random vectors $\mathbf{X}$ and $\mathbf{Y}$,

$$Var_{\mathbf{Y}}(Y_i) = Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})] + E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})].$$



Proof. $$Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{x}) = E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{x}) - [E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{x})]^2,$$

and, $$E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})] = E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\}.$$

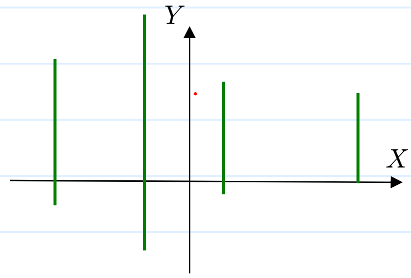
Also, $$Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})] = E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} - \{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]\}^2.$$

Now, $$\begin{aligned} Var_{\mathbf{Y}}(Y_i) &= E_{\mathbf{Y}}(Y_i^2) - [E_{\mathbf{Y}}(Y_i)]^2 \\ &= E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - \{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]\}^2 \\ &= E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i^2|\mathbf{X})] - E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} \\ &\quad + E_{\mathbf{X}}\{[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]^2\} - \{E_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]\}^2 \\ &= E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})] + Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]. \end{aligned}$$

- Corollary.
  - $Var_{\mathbf{Y}}(Y_i) \geq E_{\mathbf{X}}[Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$ and the equality holds if and only if

$$E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X}) = E_{\mathbf{Y}}(Y_i)$$

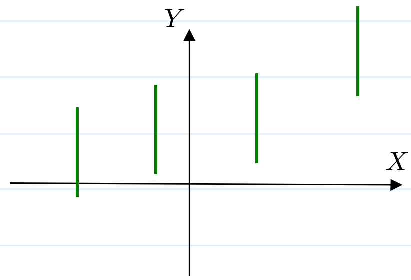
with probability one.



  - $Var_{\mathbf{Y}}(Y_i) \geq Var_{\mathbf{X}}[E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X})]$ and the equality hold if and only if

$$Var_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X}) = 0 \ (\Rightarrow Y_i = E_{\mathbf{Y}|\mathbf{X}}(Y_i|\mathbf{X}))$$

with probability one.



❖ **Reading**: textbook, Sec 7.5

Conditional Expectation and Prediction

- Problem formulation: predicting the value of a r.v. Y on the basis of the observed value of a r.v. X
  - Data: X and Y (example?)

- Statistical modeling: assigning (X, Y) a (known) joint distribution (cdf $F(x, y)$, pdf $f(x, y)$, or pmf $p(x, y)$)
- Objective: predicting Y by using a function of X, i.e.,

$$g(X) \leftarrow \text{predictor}$$

- Predictor: considering the following three groups of g's
  - (i) $G_1 = \{g(x) : g(x) = c, \text{ where } c \in \mathbb{R}\}$
  - (ii) $G_2 = \{g(x) : g(x) = a + bx, \text{ where } a, b \in \mathbb{R}\}$
  - (iii) $G_3 = \{g(x) : g \text{ is an arbitrary function}\}$

  Note. $G_1 \subset G_2 \subset G_3$.

- Question: Within each group, what is the "best" predictor?
- Criterion: minimizing mean square error

$$\text{MSE} \equiv E_{X,Y}\{[Y - g(X)]^2\}$$

- Theorem (best *constant* predictor under MSE).

$$E_{X,Y}(Y-c)^2 = E_Y(Y-c)^2 \geq E_Y[Y-E_Y(Y)]^2 = Var_Y(Y)$$

The equality holds if and only if $c=E_Y(Y)$.

Proof.

$$E_Y(Y-c)^2 = Var_Y(Y) + (\mu_Y - c)^2 \geq Var_Y(Y)$$

- Theorem (best predictor under MSE).

$$E_{X,Y}[Y-g(X)]^2 \geq E_{X,Y}[Y-E_{Y|X}(Y|X)]^2 = E_X[Var_{Y|X}(Y|X)]$$

The equality holds if and only if $g(x)=E_{Y|X}(Y|x)$.

Proof. $E_{X,Y}[Y-g(X)]^2$

$$\begin{aligned}
&= E_{X,Y}\{[Y - E_{Y|X}(Y|X)] + [E_{Y|X}(Y|X) - g(X)]\}^2 \\
&= E_{X,Y}[Y - E_{Y|X}(Y|X)]^2 + E_X[E_{Y|X}(Y|X) - g(X)]^2 \\
&\quad + 2 \cdot E_{X,Y}\{[Y - E_{Y|X}(Y|X)][E_{Y|X}(Y|X) - g(X)]\} \\
&= E_{X,Y}[Y - E_{Y|X}(Y|X)]^2 + E_X[E_{Y|X}(Y|X) - g(X)]^2 \\
&\geq E_{X,Y}[Y - E_{Y|X}(Y|X)]^2
\end{aligned}$$

where the last "=" comes from

$$\begin{aligned}
&E_{X,Y}\{[Y - E_{Y|X}(Y|X)][E_{Y|X}(Y|X) - g(X)]\} \\
&= E_X E_{Y|X}\left\{[Y - E_{Y|X}(Y|X)]\,[E_{Y|X}(Y|X) - g(X)] \,\middle|\, X\right\} \\
&= E_X\{[E_{Y|X}(Y|X) - g(X)]\; E_{Y|X}[Y - E_{Y|X}(Y|X)|X]\} = 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&E_{X,Y}[Y - E_{Y|X}(Y|X)]^2 \\
&= E_X E_{Y|X}\{[Y - E_{Y|X}(Y|X)]^2|X\} = E_X[Var_{Y|X}(Y|X)]
\end{aligned}$$

➢ Some notes for the best predictor in G_3

- $E_{Y|X}(Y|x)$ is the best predictor of Y based on X, in the sense of mean square prediction error
- Its calculation requires to know the joint distribution of X and Y, or at least $E_{Y|X}(Y|x)$
- $E_{Y|X}(Y|x)$ is called the regression function of Y on X

- Theorem (best *linear* predictor under MSE).

$$E_{X,Y}[Y-(a+bX)]^2 \geq E_{X,Y}\left\{Y-\left[\mu_Y+\rho_{XY}\frac{\sigma_Y}{\sigma_X}(X-\mu_X)\right]\right\}^2$$
$$= \sigma_Y^2(1-\rho_{XY}^2)$$

The equality holds if and only if $a=\mu_Y-b\mu_X$ and $b=\rho_{XY}\sigma_Y/\sigma_X$.

Proof. $E_{X,Y}(Y-a-bX)^2$

$$\begin{aligned}
&= Var_{X,Y}(Y-a-bX)+[E_{X,Y}(Y-a-bX)]^2 \\
&= Var_{X,Y}(Y-bX)+(\mu_Y-a-b\mu_X)^2 \\
&\geq Var_{X,Y}(Y-bX) \qquad (\Rightarrow \text{setting } a=\mu_Y-b\mu_X) \\
&= \sigma_Y^2+b^2\sigma_X^2-2b\sigma_{XY} \\
&= \sigma_X^2\left(b^2-2b\frac{\sigma_{XY}}{\sigma_X^2}+\frac{\sigma_{XY}^2}{\sigma_X^4}\right)+\sigma_Y^2-\frac{\sigma_{XY}^2}{\sigma_X^2} \\
&= \sigma_X^2\left(b-\frac{\sigma_{XY}}{\sigma_X^2}\right)^2+\sigma_Y^2(1-\rho_{XY}^2) \\
&\geq \sigma_Y^2(1-\rho_{XY}^2) \qquad \left(\Rightarrow \text{setting } b=\frac{\sigma_{XY}}{\sigma_X^2}=\frac{\sigma_{XY}}{\sigma_X\sigma_Y}\times\frac{\sigma_Y}{\sigma_X}=\rho_{XY}\frac{\sigma_Y}{\sigma_X}\right)
\end{aligned}$$

- Some notes for the best *linear* predictor in G_2
  - $E_{Y|X}(Y|x)=\mu_Y+(\rho_{XY}\sigma_Y/\sigma_X)(x-\mu_X)$ if (X, Y) is distributed as bivariate normal.
  - Its calculation requires to know the means, variances, and covariance of X and Y.
  - $\sigma_Y^2(1-\rho_{XY}^2)$ is small if ρ_{XY} is close to +1 or −1, and large if ρ_{XY} is close to 0.

- A comparison of these minimum MSEs
  - $\min_{a,b} E_{X,Y}[Y-(a+bX)]^2 \leq \min_c E_{X,Y}(Y-c)^2$ and the equality holds if and only if $\rho_{XY}=0$.
  - $\min_g E_{X,Y}[Y-g(X)]^2 \leq \min_{a,b} E_{X,Y}[Y-(a+bX)]^2$ and the equality holds if and only if $E_{Y|X}(Y|x)=\mu_Y+(\rho_{XY}\sigma_Y/\sigma_X)(x-\mu_X)$.

❖ **Reading**: textbook, Sec 7.6

Moment Generating Function

- Definition (Moment and Central Moment). If a random variable X has a cdf F_X, then

$$\mu_k \equiv E(X^k) = \int_{-\infty}^{\infty} x^k \, dF_X(x), \quad k = 1, 2, 3, \ldots,$$

are called the k^{th} *moments* of X provided that the integral converges absolutely, and

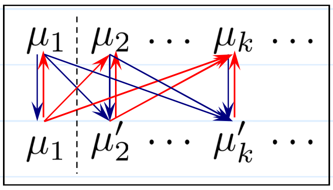
$$\mu'_k \equiv E[(X - \mu_X)^k] = \int_{-\infty}^{\infty} (x - \mu_X)^k \, dF_X(x), \quad k = 2, 3, \ldots,$$

are called k^{th} *moment about the mean* μ_X or *central moment* of X provided that the integral converges absolutely.

➢ Some notes.

- $\mu'_k = E[(X - \mu_X)^k] = E\left[\sum_{i=0}^{k} \binom{k}{i} (-\mu_X)^{k-i} X^i\right]$
  $= \sum_{i=0}^{k} \binom{k}{i} (-\mu_X)^{k-i} E(X^i) = \sum_{i=0}^{k} \binom{k}{i} (-\mu_X)^{k-i} \mu_i.$

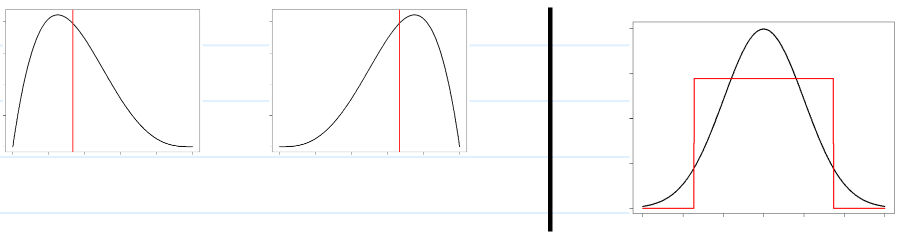
- $$\begin{aligned} \mu_k &= E(X^k) = E\{[(X - \mu_X) + \mu_X]^k\} \\ &= \sum_{i=0}^{k} \binom{k}{i} (\mu_X)^{k-i} E[(X - \mu_X)^i] \\ &= \sum_{i=0}^{k} \binom{k}{i} (\mu_X)^{k-i} \mu'_i. \end{aligned}$$



- In particular,

$$\begin{aligned} E(X) &= \mu_X = \mu_1, \quad \text{and}, \\ Var(X) &= \sigma_X^2 = \mu'_2 = \mu_2 - \mu_1^2. \end{aligned}$$

- The (central) moments give a lot of useful information about the distribution in addition to mean and variance, e.g.,
  - Skewness (a measure of the asymmetry): μ'_3/σ^3.
  - Kurtosis (a measure of the "heavy tails"): μ'_4/σ^4.



➢ Example (Uniform). If $X \sim$ Uniform(0, 1), then

$$\mu_k = \int_0^1 x^k \, dx = \frac{1}{k+1},$$

therefore,

$$\begin{aligned} \mu_X &= \mu_1 = 1/2, \quad \text{and}, \\ \sigma_X^2 &= \mu_2 - \mu_1^2 = 1/3 - (1/2)^2 = 1/12. \end{aligned}$$

And, $\mu'_k = \int_0^1 (x - 1/2)^k \, dx = \int_{-1/2}^{1/2} z^k \, dz$

$$= \frac{1}{k+1}\left[(1/2)^{k+1} - (-1/2)^{k+1}\right] = \begin{cases} 0, & k \text{ is odd}, \\ \frac{1}{(k+1)2^k}, & k \text{ is even}. \end{cases}$$

- Recall. How to characterize a distribution?

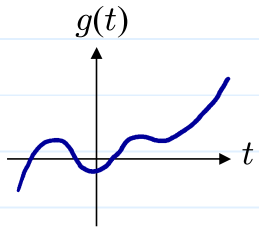
  (1) pdf/pmf, (2) cdf, (3) mgf

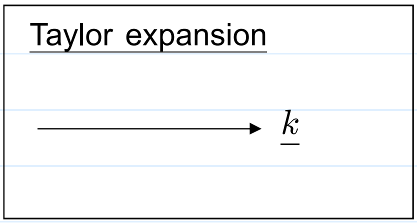
- Definition (Moment Generating Function). If X is a random variable with the cdf F_X, then

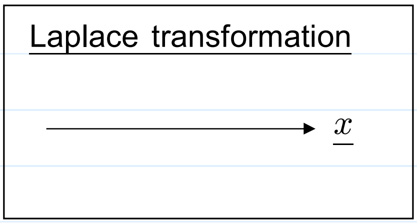
$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx}\, dF_X(x),$$

is called the *moment generating function* (mgf) of X provided that the integral converges absolutely in some non-degenerate interval of t.

$$g(t) = \sum_{k=0}^{\infty} a_k\, t^k \qquad g(t) = \int_{\mathbb{R}} f(x)\, (e^t)^x\, dx$$







➢ Some Notes.
- The mgf is a function of the variable t.
- The mgf may only exist for some particular values of t.
- $M_X(t)$ always exists at $t=0$ and $M_X(0)=1$

➢ Example.
- If X is a discrete r.v. taking on values x_i's with probability p_i's, i=1, 2, 3, …, then

$$M_X(t) = E(e^{tX}) = \sum_{i=1}^{\infty} e^{tx_i} p_i.$$

- If $X \sim$ Poisson(λ), then for $-\infty<t<\infty$,

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \times \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\left(e^{\lambda e^t}\right) \sum_{x=0}^{\infty} \frac{e^{-(\lambda e^t)}(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}.$$

- If $X \sim$ exponential(λ), then for $t<\lambda$,

$$M_X(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} \times \lambda\, e^{-\lambda x}\, dx$$

$$= \lambda\left(\frac{1}{\lambda - t}\right) \int_0^{\infty} (\lambda - t)\, e^{-(\lambda-t)x}\, dx = \frac{\lambda}{\lambda - t},$$

  and $M_X(t)$ does not exist for $t \geq \lambda$.

- A list of some mgfs (exercise)
  - If $X \sim$ binomial(n, p),

$$M_X(t) = (1 - p + pe^t)^n, \text{ for } t < -\log(1-p).$$

- If $X \sim$ negative binomial(r, p),

$$M_X(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r, \text{ for } t < -\log(1-p).$$

- If $X \sim$ uniform(α, β), $M_X(t) = \frac{e^{\beta t} - e^{\alpha t}}{t(\beta-\alpha)}$.
- If $X \sim$ gamma(α, λ),

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha, \text{ for } t < \lambda.$$

- If $X \sim$ beta(α, β), $M_X(t) = 1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r}\right) \frac{t^k}{k!}$.
- If $X \sim$ normal(μ, σ^2), $M_X(t) = e^{\mu t + (\sigma^2/2)t^2}$.

- Theorem (Uniqueness Theorem). Suppose that the mgfs $M_X(t)$ and $M_Y(t)$ of random variables X and Y exist for all $|t|<h$ for some $h>0$. If

$$M_X(t) = M_Y(t),$$

for $|t|<h$, then

$$F_X(z) = F_Y(z)$$

for all $z \in \mathbb{R}$, where F_X and F_Y are the cdfs of X and Y, respectively.

Proof. Skipped (by the uniqueness theorem of Laplace transform.)

➢ Application of the uniqueness theorem

- When a mgf exists for all $|t|<h$ for some $h>0$, there is a *unique* distribution corresponding to that mgf.
- This allows us to use mgfs to find distributions of *transformed* random variables in some cases.
- This technique is most commonly used for *linear combinations* of independent random variables $X_1, \ldots, X_n$

➢ Example. If $M_X(t) = p_1 e^{a_1 t} + \cdots + p_k e^{a_k t}$, where $p_1 + \cdots + p_k = 1$, then X is a discrete r.v. and its pmf is

$$p_X(x) = \begin{cases} p_i, & \text{for } x = a_i,\ i = 1, \ldots, k, \\ 0, & \text{otherwise.} \end{cases}$$

- Theorem (Moments and MGF). If $M_X(t)$ exists for $|t|<h$ for some $h>0$, then

$$M_X(0) = 1,$$

and,

$$M_X^{(k)}(0) = \mu_k, \quad k = 1, 2, 3, \ldots$$

Proof. First, $M_X(0) = \int_{-\infty}^{\infty} e^{0\cdot x}\, dF_X(x) = \int_{-\infty}^{\infty} 1\, dF_X(x) = 1$.

$$M_X{}'(0) = \frac{d}{dt} M_X(t)\Big|_{t=0} = \left[\frac{d}{dt}\int_{-\infty}^{\infty} e^{tx}\, dF_X(x)\right]\Big|_{t=0}$$

$$= \int_{-\infty}^{\infty}\left(\frac{d}{dt}e^{tx}\Big|_{t=0}\right) dF_X(x) = \int_{-\infty}^{\infty}\left(xe^{tx}|_{t=0}\right) dF_X(x)$$

$$= \int_{-\infty}^{\infty} x\cdot 1\, dF_X(x) = E_X(X) = \mu_1.$$

$$\cdots = \cdots$$

$$M_X^{(k)}(0) = \frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \left[\frac{d^k}{dt^k}\int_{-\infty}^{\infty} e^{tx}\, dF_X(x)\right]\Big|_{t=0}$$

$$= \int_{-\infty}^{\infty}\left(\frac{d^k}{dt^k}e^{tx}\Big|_{t=0}\right) dF_X(x) = \int_{-\infty}^{\infty}\left(x^k e^{tx}|_{t=0}\right) dF_X(x)$$

$$= \int_{-\infty}^{\infty} x^k\cdot 1\, dF_X(x) = E_X(X^k) = \mu_k.$$

- Example. If $X \sim$ exponential(λ), then $M_X(t) = \frac{\lambda}{\lambda - t}$.

  Because $$M_X^{(k)}(t) = \frac{k!\,\lambda}{(\lambda-t)^{k+1}},$$

  we get $$\mu_k = M_X^{(k)}(0) = \frac{k!}{\lambda^k}.$$

- Theorem (MGF for linear transformation). For constants a and b,

$$M_{a+bX}(t) = e^{at} M_X(bt).$$

Proof. $M_{a+bX}(t) = E_X[e^{t(a+bX)}] = e^{at}\, E_X[e^{(bt)X}] = e^{at} M_X(bt)$.

- Theorem (MGF for SUM of independent r.v.'s). If $X_1, \ldots, X_n$ are independent each with mgfs $M_{X_1}(t), \ldots, M_{X_n}(t)$, respectively, then the mgf of $S=X_1+\cdots+X_n$ is

$$M_S(t) = M_{X_1}(t) \times \cdots \times M_{X_n}(t).$$

Proof. $M_S(t) = E_S(e^{tS}) = E_{X_1,\ldots,X_n}[e^{t(X_1+\cdots+X_n)}]$

$$= E_{X_1,\ldots,X_n}(e^{tX_1} \times \cdots \times e^{tX_n})$$

$$= E_{X_1}(e^{tX_1}) \times \cdots \times E_{X_n}(e^{tX_n}) = M_{X_1}(t) \times \cdots \times M_{X_n}(t).$$

- Example. If $X_1, \ldots, X_n$ are i.i.d. $\sim$ geometric(p), then $S=X_1+\cdots+X_n \sim$ negative binomial(n, p).

  Proof. $M_S(t) = M_{X_1}(t) \times \cdots \times M_{X_n}(t)$

$$= \frac{pe^t}{1-(1-p)e^t} \times \cdots \times \frac{pe^t}{1-(1-p)e^t} = \left[\frac{pe^t}{1-(1-p)e^t}\right]^n.$$

➢ Example. If $X_1, \ldots, X_n$ are independent and

$$X_i \sim \text{normal}(\mu_i, \sigma_i^2), \text{ for } i=1, \ldots, n.$$

Let $S = a_0 + a_1 X_1 + \cdots + a_n X_n$, then

$$S \sim \text{normal}\left(a_0 + a_1\mu_1 + \cdots + a_n\mu_n,\ a_1^2\,\sigma_1^2 + \cdots + a_n^2\,\sigma_n^2\right).$$

Proof. $M_S(t) = e^{a_0 t} \times \prod_{i=1}^n e^{\mu_i(a_i t) + (\sigma_i^2/2)(a_i t)^2}$

$$= \quad e^{(a_0 + a_1\mu_1 + \cdots + a_n\mu_n)t + \left[(a_1^2\sigma_1^2 + \cdots + a_n^2\sigma_n^2)/2\right]t^2}.$$

- Definition (Joint Moment Generating Function). For random variables $X_1, \ldots, X_n$, their joint mgf is defined as

$$M_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = E_{X_1,\ldots,X_n}\left(e^{t_1 X_1 + \cdots + t_n X_n}\right)$$

provided that the expectation exists.

➢ Example. If $X_1, \ldots, X_m \sim$ multinomial$(n, m, p_1, \ldots, p_m)$, the joint pmf is:

$$\binom{n}{x_1, \ldots, x_m} p_1^{x_1} \cdots p_m^{x_m}$$

$$M_{X_1,\ldots,X_m}(t_1, \ldots, t_m)$$

$$= \sum_{\substack{0 \le x_i \le n,\ i=1,\ldots,m \\ x_1 + \cdots + x_m = n}} e^{t_1 x_1 + \cdots + t_m x_m} \binom{n}{x_1, \ldots, x_m} p_1^{x_1} \cdots p_m^{x_m}$$

$$= \sum_{\substack{0 \le x_i \le n,\ i=1,\ldots,m \\ x_1 + \cdots + x_m = n}} (p_1\, e^{t_1})^{x_1} \cdots (p_m\, e^{t_m})^{x_m}$$

$$= \left(p_1 e^{t_1} + \cdots + p_m e^{t_m}\right)^n.$$

- Some Properties of Joint mgf
  - ➢ $M_{X_1}(t) = M_{X_1, X_2, \ldots, X_n}(t, 0, \ldots, 0)$.
  - ➢ uniqueness theorem
  - ➢ $X_1, \ldots, X_n$ are independent if and only if

$$M_{X_1,\ldots,X_n}(t_1, \ldots, t_n) = M_{X_1}(t_1) \times \cdots \times M_{X_n}(t_n).$$

  - ➢ $\frac{\partial^{k_1 + \cdots + k_n}}{\partial t_1^{k_1} \cdots \partial t_n^{k_n}} M_{X_1,\ldots,X_n}(0, \ldots, 0) = E_{X_1,\ldots,X_n}(X_1^{k_1} \times \cdots \times X_n^{k_n})$.

❖ **Reading**: textbook, Sec 7.7