➢ Example. If X and Y have a joint pdf

$$f(x,y) = \frac{2}{(1+x+y)^3},$$

examine whether it's a joint pdf (exercise)

for $0 \le x,\ y < \infty$, then

$$f_X(x) = \int_0^\infty f(x,y)\,dy = -\frac{1}{(1+x+y)^2}\bigg|_0^\infty = \frac{1}{(1+x)^2},$$

for $0 \le x < \infty$. So,

$$\underbrace{f_{Y|X}(y|x)}_{\text{random}} \overset{\text{fixed}}{=} \frac{f(x,y)}{f_X(x)} = \frac{2(1+x)^2}{(1+x+y)^3},$$

and, $P(Y > c \,|\, X = x) = \int_c^\infty \frac{2(1+x)^2}{(1+x+y)^3}\,dy$

$X \in x \pm \frac{\Delta x}{2}$

$$= -\frac{(1+x)^2}{(1+x+y)^2}\bigg|_{y=c}^\infty = \frac{(1+x)^2}{(1+x+c)^2}.$$

a function of x

joint / marginal

- Mixed Joint Distribution: Definition of conditional distribution can be similarly generalized to the case in which some random variables are discrete and the others continuous (see a later example).

Recall. The 3 laws in LNp.4-11~13 | $P(A_1\cap\cdots\cap A_n) = P(A_1)\cdot P(A_2|A_1)\cdot\cdots\cdot P(A_n|A_1\cap\cdots\cap A_{n-1})$

- Theorem (Multiplication Law). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X}, \mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$, then

$P(\mathbf{X}=\mathbf{x} \cap \mathbf{Y}=\mathbf{y})$

$$p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) \overset{\text{cf. independent}}{=} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \times p_{\mathbf{X}}(\mathbf{x}), \text{ or}$$

$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \times f_{\mathbf{X}}(\mathbf{x}).$$

similar extension

$p_{X_1,\dots,X_n}(x_1,\dots,x_n) = p_{X_1}(x_1) \times p_{X_2|X_1}(x_2|x_1) \times \cdots \times p_{X_n|X_1,\dots,X_{n-1}}(x_n|x_1,\dots,x_{n-1})$

Proof. By the definition of conditional distribution.

pmf (LNp.7-51), pdf (LNp.7-53)

$P(B) = \sum_i P(B|A_i)P(A_i)$ partition

- Theorem (Law of Total Probability). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X}, \mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$, then

$\sum_{\mathbf{x}\in\mathcal{X}} p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$

$$p_{\mathbf{Y}}(\mathbf{y}) = \sum_{\mathbf{x}=-\infty}^{\infty} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})p_{\mathbf{X}}(\mathbf{x}), \text{ or}$$

$\sum_{\mathbf{x}\in\mathcal{X}}$

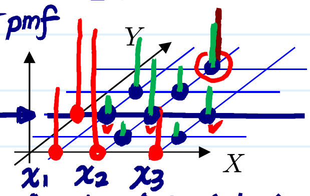
$A_{\mathbf{x}} = \{\mathbf{X}=\mathbf{x}\},\ \mathbf{x}\in\mathcal{X}$ form a partition of Ω

$P(\mathbf{Y}=\mathbf{y})$

$$f_{\mathbf{Y}}(\mathbf{y}) = \int_{-\infty}^{\infty} f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x}.$$

$\int_{\mathbb{R}^n}$

joint pmf



partition: $P(A_i) \xrightarrow{\text{update}} P(A_i|B)$

Proof. By the definition of marginal distribution and the multiplication law.

y, $p_Y(y)$, x_1, x_2, x_3, $p_{Y|X}(y|x_1)$, $p_{Y|X}(y|x_2)$, $p_{Y|X}(y|x_3)$

- Theorem (Bayes Theorem). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X}, \mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$, then

joint / marginal

update

$$p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) = \frac{p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})p_{\mathbf{X}}(\mathbf{x})}{\sum_{\mathbf{x}'=-\infty}^{\infty} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}')p_{\mathbf{X}}(\mathbf{x}')}, \text{ or}$$

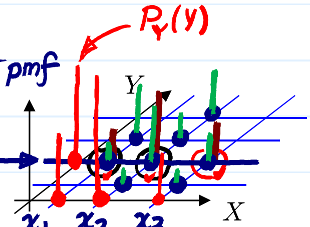
meaning

$\frac{p_{X,Y}(x,y)}{p_Y(y)}$

$$f_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) = \frac{f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})f_{\mathbf{X}}(\mathbf{x})}{\int_{-\infty}^{\infty} f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}')f_{\mathbf{X}}(\mathbf{x}')\,d\mathbf{x}'}.$$

joint pmf

$p_Y(y)$



$p_Y(y)$, y, $p_{X|Y}(x|y)$, x_1, x_2, x_3, $p_{Y|X}(y|x_3)$, $p_{Y|X}(y|x_2)$, $p_{Y|X}(y|x_1)$

Proof. By the definition of conditional distribution, multiplication law, and the law of total probability.

Bayesian estimation

Example.

Gold coins example in LNp.4-29 cf.

X, 1st, ? x, 2nd, B, (LNp.4-14)

before, after, now

perform indep Bernoulli trials $(Y_1, \ldots, Y_n)$ depending on x; $(y_1, \ldots, y_n)$

- Suppose that $X \sim$ Uniform(0, 1), and $(Y_1, \ldots, Y_n|X=x)$ are i.i.d. with Bernoulli(x), i.e.,

conditional independent — Q: $Y_1, \ldots, Y_n$ indep.? NO

Note. X: continuous, $Y_1, \ldots, Y_n$: discrete

$$p_{\mathbf{Y}|X}(y_1, \ldots, y_n|x) = x^{y_1+\cdots+y_n}(1-x)^{n-(y_1+\cdots+y_n)},$$

fixed

for $y_1, \ldots, y_n \in \{0, 1\}$.

after 1st stage / before 1st stage cf.

joint dist.: a mix of pmf & pdf

- By the multiplication law, for $y_1, \ldots, y_n \in \{0, 1\}$ and $0<x<1$,

$X \in x \pm (\Delta x/2)$

$$p_{\mathbf{Y},X}(y_1, \ldots, y_n, x) = x^{y_1+\cdots+y_n}(1-x)^{n-(y_1+\cdots+y_n)} \times \Delta x$$

random

Intuition. What values of x are more probable?

- Suppose that we observed $Y_1=1, \ldots, Y_n=1$. (or $Y_1=y_1, \ldots, Y_n=y_n$)

now; k "1", $n-k$ "0" $\Rightarrow k = \sum_{i=1}^n y_i$

- By the law of total probability,

marginal dist. of $Y_1, \ldots, Y_n$

$$P(Y_1=1, \ldots, Y_n=1) = p_{\mathbf{Y}}(1, \ldots, 1)$$
$$= \int_0^1 p_{\mathbf{Y}|X}(1, \ldots, 1|x) f_X(x)\, dx$$
$$= \int_0^1 x^n\, dx = \frac{1}{n+1}x^{n+1}\Big|_0^1 = \frac{1}{n+1}.$$

(exercise)

$$p_{\mathbf{Y}}(y_1, \ldots, y_n) = \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} = \frac{k!(n-k)!}{(n+1)!} = \frac{(\sum_{i=1}^n y_i)!(n-\sum_{i=1}^n y_i)!}{(n+1)!}$$

- And, by Bayes' Theorem,

joint / marginal

$$f_{X|\mathbf{Y}}(x|Y_1=1, \ldots, Y_n=1) = \frac{p_{\mathbf{Y}|X}(1, \ldots, 1|x) f_X(x)}{p_{\mathbf{Y}}(1, \ldots, 1)} = (n+1)x^n.$$

now; before; update; after

$$\frac{\Gamma(n+2)}{\Gamma(n+1)\Gamma(1)} x^{(n+1)-1}(1-x)^{1-1}$$

$$p_{\mathbf{Y}}(1, \ldots, 1) = \int_0^1 p_{\mathbf{Y}|X}(1, \ldots, 1|x') f_X(x')\,dx'$$

$E(X|Y_1=1, \ldots, Y_n=1) = \frac{n+1}{n+2}$

for $0<x<1$, i.e., $(X|Y_1=1, \ldots, Y_n=1) \sim$ Beta($n+1$, 1). update

Why is X still random?

(cf., marginal distribution of $X \sim$ Uniform(0, 1)=Beta(1, 1).)

- If there were an $(n+1)^{\text{st}}$ Bernoulli trial Y_{n+1},

probability of getting "1" after we know $Y_1=Y_2=\cdots=Y_n=1$

$$P(Y_{n+1}=1|Y_1=1, \ldots, Y_n=1)$$
$$= \frac{P(Y_1=1, \ldots, Y_{n+1}=1)}{P(Y_1=1, \ldots, Y_n=1)} = \frac{1/(n+2)}{1/(n+1)} = \boxed{\frac{n+1}{n+2}}.$$

Why not x?

- $Y_{n+1}|Y_1=y_1, \ldots, Y_n=y_n \sim$ Bernoulli$\left(\frac{\sum_{i=1}^n y_i+1}{n+2}\right)$ (exercise) cf.
- $Y_{n+1} \sim$ Bernoulli($\frac{1}{2}$) (exercise)
- $Y_{n+1}|X=x \sim$ Bernoulli(x)

- (exercise) In general, it can be shown that

$(X|Y_1=y_1, \ldots, Y_n=y_n) \sim$ Beta($(y_1+\cdots+y_n)+1, n-(y_1+\cdots+y_n)+1$).

$\to E(X|Y_1=y_1, \ldots, Y_n=y_n) = (\sum_{i=1}^n y_i+1)/(n+2)$ ← a Bayesian estimator of x.

- Theorem (Conditional Distribution & Independent). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X}, \mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$. Then, $\mathbf{X}$ and $\mathbf{Y}$ are independent, i.e.,

$$P_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \cdot P_{\mathbf{X}}(\mathbf{x}) = p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = p_{\mathbf{X}}(\mathbf{x}) \times p_{\mathbf{Y}}(\mathbf{y}), \quad \text{or}$$

$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \times f_{\mathbf{Y}}(\mathbf{y}),$$

if and only if

$$\frac{P_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{P_{\mathbf{X}}(\mathbf{x})} = p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = p_{\mathbf{Y}}(\mathbf{y}), \quad \text{or}$$

$$f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = f_{\mathbf{Y}}(\mathbf{y}).$$

use the concept to examine whether the $(Y_1, \cdots, Y_n)$ in LNp.7-58 are indep.?

Proof. By the definition of conditional distribution.

➢ Intuition.

- The 2 graphs about the joint pmf/pdf of independent r.v.'s in LNp.7-27
- $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ or $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ offers information about the distribution of **Y** when **X**=**x**.

  $p_{\mathbf{Y}}(\mathbf{y})$ or $f_{\mathbf{Y}}(\mathbf{y})$ offers information about the distribution of **Y** when **X** not observed.

cf.

❖ **Reading**: textbook, Sec 6.4, 6.5