➢ Example. If X and Y have a joint pdf

$$f(x,y)=\frac{2}{(1+x+y)^3},$$

examine whether it's a joint pdf (exercise)

for $0\le x,\ y<\infty$, then

$$f_X(x)=\int_0^\infty f(x,y)\,dy=-\frac{1}{(1+x+y)^2}\Big|_0^\infty=\frac{1}{(1+x)^2},$$

for $0\le x<\infty$. So,

$$\underset{\text{random}}{f_{Y|X}(y|\overset{\text{fixed}}{x})}=\frac{f(x,y)}{f_X(x)}=\frac{2(1+x)^2}{(1+x+y)^3},$$

joint / marginal

and, $P(Y>c|X=x)$ ($X\in x\pm\frac{\Delta x}{2}$)

$$=\int_c^\infty\frac{2(1+x)^2}{(1+x+y)^3}\,dy=-\frac{(1+x)^2}{(1+x+y)^2}\Big|_{y=c}^\infty=\frac{(1+x)^2}{(1+x+c)^2}.$$

a function of x

- Mixed Joint Distribution: Definition of conditional distribution can be similarly generalized to the case in which some random variables are discrete and the others continuous (see a later example).

Recall. The 3 laws in LNp.4-11~13. $P(A_1\cap\cdots\cap A_n)=P(A_1)\cdot P(A_2|A_1)\cdots\cdot P(A_n|A_1\cap\cdots\cap A_{n-1})$

- Theorem (Multiplication Law). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X},\mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$, then

$$P(\mathbf{X}=\mathbf{x}\cap\mathbf{Y}=\mathbf{y})\leftarrow p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})\overset{\text{cf. independent}}{=}p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\times p_{\mathbf{X}}(\mathbf{x}),\ \text{or}$$

$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})=f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\times f_{\mathbf{X}}(\mathbf{x}).$$

similar extension: $p_{X_1,\dots,X_n}(x_1,\dots,x_n)=p_{X_1}(x_1)\times p_{X_2|X_1}(x_2|x_1)\times\cdots\times p_{X_n|X_1\cdots X_{n-1}}(x_n|x_1,\dots,x_{n-1})$

Proof. By the definition of conditional distribution.
↳ pmf (LNp.7-51), pdf (LNp.7-53)

$P(B)=\sum_i P(B|A_i)P(A_i)$ partition

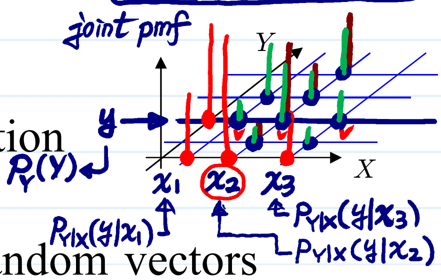
- Theorem (Law of Total Probability). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X},\mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$, then

$$\boxed{\sum_{x\in\mathcal{X}}p_{X,Y}(x,y)}\ p_{\mathbf{Y}}(\mathbf{y})=\sum_{\mathbf{x}=-\infty}^{\infty}p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})p_{\mathbf{X}}(\mathbf{x}),\ \text{or}$$

$\sum_{x\in\mathcal{X}}$; $A_x=\{\mathbf{X}=x\},\ x\in\mathcal{X}$ form a partition of Ω

$$P(\mathbf{Y}=\mathbf{y})\quad f_{\mathbf{Y}}(\mathbf{y})=\int_{-\infty}^{\infty}f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x}.$$

$\int_{\mathbb{R}^n}$



partition: $P(A_i)\xrightarrow{\text{update}}P(A_i|B)$

Proof. By the definition of marginal distribution and the multiplication law.

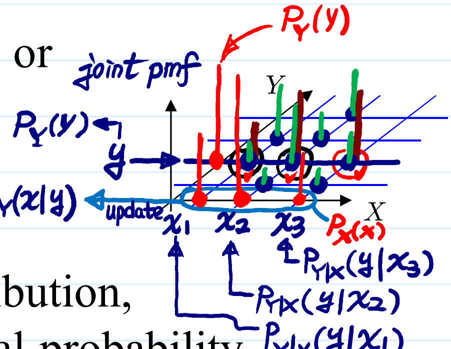
- Theorem (Bayes Theorem). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X},\mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$, then

joint / marginal

$$p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})\overset{\text{update}}{=}\frac{p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})p_{\mathbf{X}}(\mathbf{x})}{\sum_{\mathbf{x}'=-\infty}^{\infty}p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}')p_{\mathbf{X}}(\mathbf{x}')},\ \text{or}$$

meaning

$$\frac{P_{X,Y}(x,y)}{P_Y(y)}\quad f_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})=\frac{f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})f_{\mathbf{X}}(\mathbf{x})}{\int_{-\infty}^{\infty}f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}')f_{\mathbf{X}}(\mathbf{x}')\,d\mathbf{x}'}.$$



Proof. By the definition of conditional distribution, multiplication law, and the law of total probability.

Bayesian estimation → Example.

Gold coins example in LNp.4-29 (cf.)

before → 1st (X) → ? (x) → 2nd → B → (LNp.4-14)

after; now; perform indep. Bernoulli trials $(Y_1,\cdots,Y_n)$ depending on x; $(y_1,\cdots,y_n)$

Note. X: continuous, $Y_1,\cdots,Y_n$: discrete

- Suppose that $X \sim$ Uniform(0, 1), and $(Y_1, \ldots, Y_n|X=x)$ are i.i.d. with Bernoulli(x), i.e.,

conditional independent — Q: $Y_1,\cdots,Y_n$ indep.? NO

$$p_{\mathbf{Y}|X}(y_1,\ldots,y_n|x) = x^{y_1+\cdots+y_n}(1-x)^{n-(y_1+\cdots+y_n)},$$

fixed

for $y_1, \ldots, y_n \in \{0, 1\}$.

after 1st stage / before 1st stage (cf.)

joint dist.: a mix of pmf & pdf

- By the multiplication law, for $y_1, \ldots, y_n \in \{0, 1\}$ and $0<x<1$,

$X\in x\pm(\Delta x/2)$

$$p_{\mathbf{Y},X}(y_1,\ldots,y_n,x) = x^{y_1+\cdots+y_n}(1-x)^{n-(y_1+\cdots+y_n)} \times \Delta x$$

random

Intuition. What values of x are more probable?

- Suppose that we observed $Y_1=1, \ldots, Y_n=1$. (or $Y_1=y_1,\cdots,Y_n=y_n$)

now; k "1", $n-k$ "0" $\Rightarrow k=\sum_{i=1}^n y_i$.

- By the law of total probability,

$$P(Y_1=1,\ldots,Y_n=1) = p_{\mathbf{Y}}(1,\ldots,1)$$
$$= \int_0^1 p_{\mathbf{Y}|X}(1,\ldots,1|x) f_X(x)\,dx$$
$$= \int_0^1 x^n\,dx = \frac{1}{n+1}x^{n+1}\Big|_0^1 = \frac{1}{n+1}.$$

marginal dist. of $Y_1,\cdots,Y_n$

$$p_{\mathbf{Y}}(y_1,\cdots,y_n) \overset{\text{(exercise)}}{=} \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} = \frac{k!(n-k)!}{(n+1)!} = \frac{(\sum_{i=1}^n y_i)!(n-\sum_{i=1}^n y_i)!}{(n+1)!}$$

joint / marginal

- And, by Bayes' Theorem,

$$f_{X|\mathbf{Y}}(x|Y_1=1,\ldots,Y_n=1) = \frac{p_{\mathbf{Y}|X}(1,\ldots,1|x)\,\boxed{f_X(x)}}{p_{\mathbf{Y}}(1,\ldots,1)} = (n+1)x^n.$$

now; before; update; after

$$\frac{\Gamma(n+2)}{\Gamma(n+1)\Gamma(1)}x^{(n+1)-1}(1-x)^{1-1}$$

$$p_{\mathbf{Y}}(1,\ldots,1) = \int_0^1 p_{\mathbf{Y}|X}(1,\cdots,1|x')f_X(x')\,dx'$$

$E(X|Y_1=1,\cdots,Y_n=1) = \frac{n+1}{n+2}$

for $0<x<1$, i.e., $(X|Y_1=1, \ldots, Y_n=1) \sim$ Beta($n+1$, 1). (α, β) update

Why is X still random?

(cf., marginal distribution of $X \sim$ Uniform(0, 1)=Beta(1, 1).)

- If there were an $(n+1)^{st}$ Bernoulli trial Y_{n+1},

probability of getting "1" after we know $Y_1=Y_2=\cdots=Y_n=1$

$$P(Y_{n+1}=1|Y_1=1,\ldots,Y_n=1)$$
$$= \frac{P(Y_1=1,\ldots,Y_{n+1}=1)}{P(Y_1=1,\ldots,Y_n=1)} = \frac{1/(n+2)}{1/(n+1)} = \boxed{\frac{n+1}{n+2}}.$$

- $Y_{n+1}|Y_1=y_1,\cdots,Y_n=y_n \sim \text{Bernoulli}\left(\frac{\sum_{i=1}^n y_i+1}{n+2}\right)$ (exercise) (cf.)
- $Y_{n+1} \sim \text{Bernoulli}(\frac{1}{2})$ (exercise)
- $Y_{n+1}|X=x \sim \text{Bernoulli}(x)$

- (exercise) In general, it can be shown that

Why not x?

$$(X|Y_1=y_1, \ldots, Y_n=y_n) \sim \text{Beta}((y_1+\cdots+y_n)+1, n-(y_1+\cdots+y_n)+1).$$

$\rightarrow E(X|Y_1=y_1,\cdots Y_n=y_n) = (\sum_{i=1}^n y_i+1)/(n+2)$ ← a Bayesian estimator of x.

- Theorem (Conditional Distribution & Independent). Let **X** and **Y** be random vectors and (**X**, **Y**) have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$. Then, **X** and **Y** are independent, i.e.,

$$p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \times p_{\mathbf{X}}(\mathbf{x}) = p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = p_{\mathbf{X}}(\mathbf{x}) \times p_{\mathbf{Y}}(\mathbf{y}), \quad \text{or}$$

$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \times f_{\mathbf{Y}}(\mathbf{y}),$$

if and only if

$$\frac{p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{p_{\mathbf{X}}(\mathbf{x})} = p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = p_{\mathbf{Y}}(\mathbf{y}), \quad \text{or}$$

$$f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = f_{\mathbf{Y}}(\mathbf{y}).$$

use the concept to examine whether the $(Y_1, \cdots, Y_n)$ in LNp.7-58 are indep.?

Proof. By the definition of conditional distribution.

conditionally indep: $Y_1, \cdots, Y_n | X = x$ indep. $\forall x$ $\Leftarrow\!\!\!\!\times$ mutually indep: $Y_1, \cdots, Y_n$ indep.

➢ Intuition.

- The 2 graphs about the joint pmf/pdf of independent r.v.'s in LNp.7-27
- $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ or $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ offers information about the distribution of $\mathbf{Y}$ when $\mathbf{X}=\mathbf{x}$.

  $p_{\mathbf{Y}}(\mathbf{y})$ or $f_{\mathbf{Y}}(\mathbf{y})$ offers information about the distribution of $\mathbf{Y}$ when $\mathbf{X}$ not observed.

  cf.

❖ **Reading**: textbook, Sec 6.4, 6.5