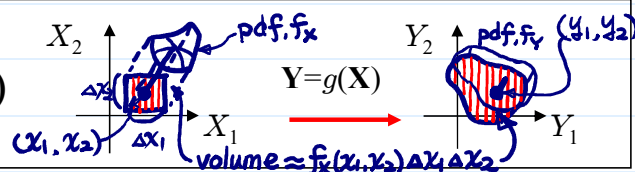


Then $f_Y(y) = f_X(g^{-1}(y)) \times |J|$, ^{cf. Thm in LNp.6-10~11}
 for y s.t. $y=g(x)$ for some x and $f_Y(y)=0$, otherwise.

Recall. The question in LNp.6-11

(Q: What is the role of $|J|$?)

why absolute value?



Proof. $F_Y(y_1, \dots, y_n) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} f_Y(t_1, \dots, t_n) dt_n \cdots dt_1$
 $= \int \cdots \int_{\substack{(x_1, \dots, x_n): \\ Y_i = g_i(x_1, \dots, x_n) \leq y_i \\ Y_n = g_n(x_1, \dots, x_n) \leq y_n}} f_X(x_1, \dots, x_n) dx_n \cdots dx_1.$

method of cdf
check 1~3 in
LNp.7-31

$X = g^{-1}(Y)$

$\int \cdots \int_S f_X(x) dx$
 $= \int \cdots \int_{g(S)} f_X(g^{-1}(y)) |J(Y)| dy$
 $\int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n}$

It then follows from an exercise in advanced calculus that

$f_Y(y_1, \dots, y_n) = \frac{\partial^n}{\partial y_1 \cdots \partial y_n} F_Y(y_1, \dots, y_n)$ ^{change of variables}
 $= f_X(g^{-1}(y)) \times |J|$ ^{Functions of X}

Remark. When the dimensionality of Y (denoted by k) is less than n , we can choose another $n-k$ transformations Z such that

$(Y, Z) = g(X)$ ^{k -dim $\rightarrow$ $(n-k)$ -dim}
 $\mathbb{R}^n \rightarrow \mathbb{R}^n, g^{-1}$ exists, g^{-1} continuously differentiable

satisfy the assumptions in above theorem.

By integrating out the last $n-k$ arguments in the joint pdf of (Y, Z) , the joint pdf of Y can be obtained.

Example. X_1 and X_2 are random variables with joint pdf $f_X(x_1, x_2)$. Find the distribution of $Y_1 = X_1 / (X_1 + X_2) \equiv g_1(X_1, X_2)$

X_1, X_2 : Continuous r.v.'s

Let $Y_2 = X_1 + X_2$, then

proportion when $X_1 \geq 0, X_2 \geq 0$

$g = (g_1, g_2)$
 $:\mathbb{R}^2 \rightarrow \mathbb{R}^2$

$g_2(X_1, X_2)$
add one more transformation

$x_1 = y_1 y_2 \equiv w_1(y_1, y_2)$
 $x_2 = y_2 - y_1 y_2 \equiv w_2(y_1, y_2)$ ^{g^{-1}}

Since $\frac{\partial w_1}{\partial y_1} = y_2, \frac{\partial w_1}{\partial y_2} = y_1, \frac{\partial w_2}{\partial y_1} = -y_2, \frac{\partial w_2}{\partial y_2} = 1 - y_1,$

$J = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{vmatrix} = y_2 - y_1 y_2 + y_1 y_2 = y_2$, and $|J| = |y_2|$. ^{determinant absolute value}

Therefore, $f_Y(y_1, y_2) = f_X(y_1 y_2, y_2 - y_1 y_2) |y_2|$,

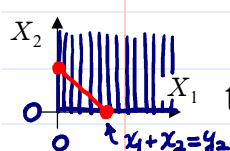
and, $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_Y(y_1, y_2) dy_2$

$= \int_{-\infty}^{\infty} f_X(y_1 y_2, y_2 - y_1 y_2) |y_2| dy_2.$

$f_X(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$ ⁽⁼⁾ $\int_{-\infty}^{\infty} f_{X_1}(y_1 y_2) f_{X_2}(y_2 - y_1 y_2) |y_2| dy_2$
 when X_1 and X_2 are independent

- Theorem. If X_1 and X_2 are independent, and

$$X_1 \sim \text{Gamma}(\alpha_1, \lambda), \quad X_2 \sim \text{Gamma}(\alpha_2, \lambda),$$



then $Y_1 = X_1/(X_1+X_2) \sim \text{Beta}(\alpha_1, \alpha_2)$.

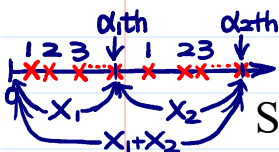
Corollary.

$X_1 \sim \text{exponential}(\lambda)$
 $X_2 \sim \text{exponential}(\lambda)$
 $\Rightarrow Y_1 \sim \text{Uniform}(0,1)$

Proof. For $x_1, x_2 \geq 0$, the joint pdf of $\underline{X}$ is

$$f_{\underline{X}}(x_1, x_2) = \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} x_1^{\alpha_1-1} e^{-\lambda x_1} \times \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} x_2^{\alpha_2-1} e^{-\lambda x_2}$$

$$= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x_1^{\alpha_1-1} x_2^{\alpha_2-1} e^{-\lambda(x_1+x_2)}.$$



So, for $0 \leq y_1 \leq 1$,

$$f_{Y_1, Y_2}(y_1, y_2) \propto g_1(y_1)g_2(y_2)$$

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1}(y_1 y_2) f_{X_2}(y_2 - y_1 y_2) |y_2| dy_2$$

$$= \int_0^{\infty} \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (y_1 y_2)^{\alpha_1-1} (y_2 - y_1 y_2)^{\alpha_2-1} e^{-\lambda y_2} \cdot y_2 dy_2$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1} (1-y_1)^{\alpha_2-1}$$

$$\times \int_0^{\infty} \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} y_2^{(\alpha_1+\alpha_2)-1} e^{-\lambda y_2} dy_2$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1} (1-y_1)^{\alpha_2-1}$$

$$E(Y_1) = \frac{\alpha_1}{\alpha_1+\alpha_2}$$

pdf of
Beta(α_1, α_2)

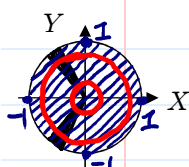
pdf of Y_2 , i.e.
 $Y_2 \sim \text{Gamma}(\alpha_1+\alpha_2, \lambda)$
(LNp.7-32)

pdf of
Gamma($\alpha_1+\alpha_2, \lambda$)

and $f_{Y_1}(y_1) = 0$, otherwise.

(exercise) Y_1 & Y_2 are independent

- Example. Suppose that X and Y have a uniform distribution over the region $D = \{(x, y): x^2 + y^2 \leq 1\}$, i.e., their joint pdf is



X, Y not
independent
(LNp.7-19)

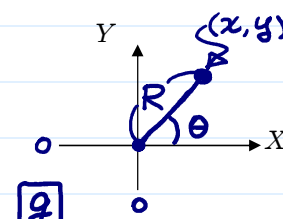
$$f_{X,Y}(x, y) = \frac{1}{\pi} \mathbf{1}_D(x, y).$$

LNp.7-18

Find the joint distribution of (R, Θ) and examine whether R and Θ are independent, where (R, Θ) is the polar coordinate representation of (X, Y) , i.e.,

$$X = R \cos(\Theta) \equiv w_1(R, \Theta),$$

$$Y = R \sin(\Theta) \equiv w_2(R, \Theta).$$



Since

$$\frac{\partial w_1}{\partial r} = \cos(\theta),$$

$$\frac{\partial w_1}{\partial \theta} = -r \sin(\theta),$$

$$\frac{\partial w_2}{\partial r} = \sin(\theta),$$

$$\frac{\partial w_2}{\partial \theta} = r \cos(\theta),$$

$$R = \sqrt{x^2 + y^2}$$

$$\Theta = \tan^{-1}\left(\frac{y}{x}\right)$$

marginal dist.

$$\Theta \sim \text{Uniform}(0, 2\pi)$$

$$R \sim \text{pdf: } 2r \mathbf{1}_{(0,1)}(r)$$

Why?

$$J = \begin{vmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{vmatrix} = r \cos^2(\theta) + r \sin^2(\theta) = r,$$

and $|J| = |r| = r$. Recall. In calculus, for polar transformation, $dx dy = r dr d\theta$

For $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$, the joint pdf of (R, Θ) is

$$f_{R,\Theta}(r, \theta) = f_{X,Y}(r \cos(\theta), r \sin(\theta)) \times |J| = \frac{1}{\pi} r = \frac{1}{2\pi} (2r)$$

and $f_{R,\Theta}(r, \theta) = 0$, otherwise.

cross
product
set

□ By the theorem in LNp.7-25, (R, Θ) are independent.

- Example. Let $X_1, \dots, X_n$ be independent and identically distributed (i.e., i.i.d.) exponential(λ). Let

$$Y_i = X_1 + \dots + X_i, \quad i = 1, \dots, n.$$

$$\begin{cases} Y_1 = X_1 \\ Y_2 = X_1 + X_2 \\ \vdots \\ Y_n = X_1 + X_2 + \dots + X_n \end{cases}$$

Find the distribution of $\mathbf{Y} = (Y_1, \dots, Y_n)$.

[Note. It has been shown that $Y_i \sim \text{Gamma}(i, \lambda)$, $i = 1, \dots, n$.]

□ The joint pdf of $X_1, \dots, X_n$ is

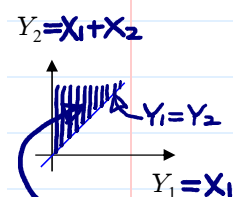
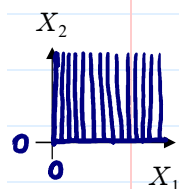
$$\begin{aligned} f_{\mathbf{X}}(x_1, \dots, x_n) &= \prod_{i=1}^n f_{X_i}(x_i) \\ &= \prod_{i=1}^n (\lambda e^{-\lambda x_i}) = \lambda^n e^{-\lambda(x_1 + \dots + x_n)} \end{aligned}$$

for $0 \leq x_i < \infty$, $i = 1, \dots, n$.

$$\begin{aligned} \text{Since } x_1 &= y_1 \equiv w_1(y_1, \dots, y_n), \\ x_2 &= y_2 - y_1 \equiv w_2(y_1, \dots, y_n), \\ &\vdots \\ x_n &= y_n - y_{n-1} \equiv w_n(y_1, \dots, y_n), \end{aligned}$$

we have

$$\frac{\partial w_i}{\partial y_j} = \begin{cases} 1, & \text{if } j = i, \\ -1, & \text{if } j = i - 1, \\ 0, & \text{otherwise,} \end{cases}$$



not a cross product set
 $Y_1 < Y_2 < \dots < Y_n$

common marginal dist.

marginal (Q: $Y_1, \dots, Y_n$ indep?)
Note: $Y_1 < Y_2 < \dots < Y_n$

LNp. 7-33

Note: not "X"

Note: not "nx"

g^{-1}

$\mathbf{Y} = g(\mathbf{X})$
 g is one-to-one
 $\Rightarrow g^{-1}$ exists

$$J = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} = 1, \text{ and } |J| = 1.$$

□ For $0 \leq y_1 \leq y_2 \leq \dots \leq y_{i-1} \leq y_i \leq y_{i+1} \leq \dots \leq y_n < \infty$,

$$\begin{aligned} f_{\mathbf{Y}}(y_1, \dots, y_n) &= f_{\mathbf{X}}(y_1, y_2 - y_1, \dots, y_n - y_{n-1}) \times |J| \\ &= \lambda^n e^{-\lambda y_n}. \end{aligned}$$

and $f_{\mathbf{Y}}(y_1, \dots, y_n) = 0$, otherwise.

Q: Are $Y_1, \dots, Y_n$ indep. by the Thm in LNp.7-25?

check * in LNp.7-41

□ The marginal pdf of Y_i is

$$f_{Y_i}(y)$$

By the Thm in LNp. 7-10

$$\begin{aligned} &\int_0^y \int_{y_1}^y \dots \int_{y_{i-2}}^y \int_y^\infty \int_{y_{i+1}}^\infty \dots \int_{y_{n-1}}^\infty \\ &\quad \lambda^n e^{-\lambda y_n} dy_n \dots dy_{y_{i+2}} dy_{i+1} dy_{i-1} \dots dy_2 dy_1 \\ &= \int_0^y \int_{y_1}^y \dots \int_{y_{i-2}}^y \lambda^i e^{-\lambda y} dy_{i-1} \dots dy_2 dy_1 \\ &= \lambda^i e^{-\lambda y} \frac{y^{i-1}}{(i-1)!} \end{aligned}$$

a constant

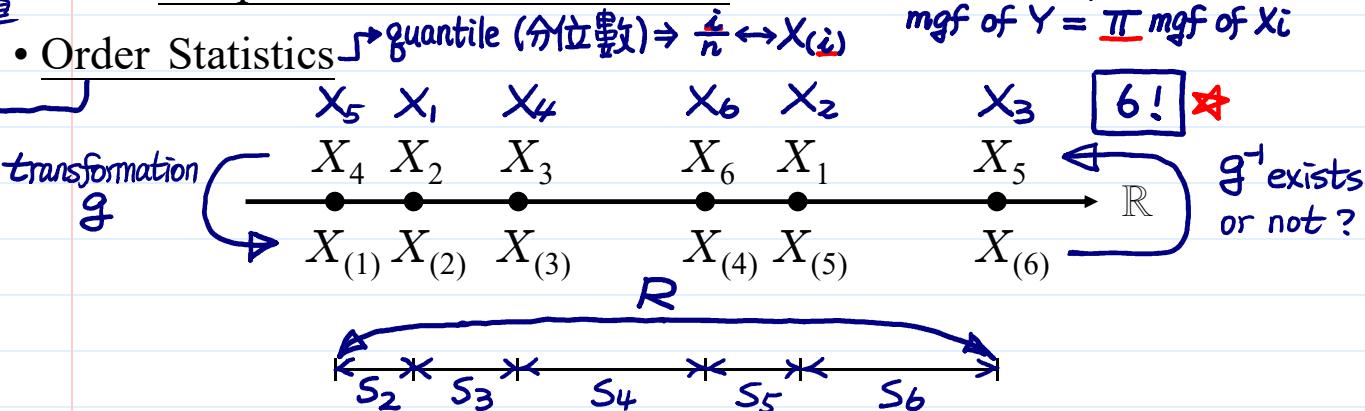
$\Gamma(i)$

pdf of $\text{Gamma}(i, \lambda)$

for $y \geq 0$, and $f_{Y_i}(y) = 0$, otherwise.

➤ Method of moment generating function.

- Based on the uniqueness theorem of moment generating function to be explained later in Chapter 7
- Especially useful to identify the distribution of sum of independent random variables. $\rightarrow X_1, \dots, X_n$ indep., $Y = X_1 + \dots + X_n$



➤ Definition. Let $X_1, \dots, X_n$ be random variables. We sort the X_i 's and denote by

$$X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$$

the order statistics. Using the notation,

$X_{(i)}$ = i th-smallest value in $X_1, \dots, X_n$, $i=1, 2, \dots, n$,

$X_{(1)} = \min(X_1, \dots, X_n)$ is the minimum,

$X_{(n)} = \max(X_1, \dots, X_n)$ is the maximum,

$R \equiv X_{(n)} - X_{(1)}$ is called range,

$S_j \equiv X_{(j)} - X_{(j-1)}$, $j=2, \dots, n$, are called j th spacing.

transformations of order stat. $X_{(1)}, \dots, X_{(n)}$

e.g.
 $X_1, \dots, X_n \sim \text{Uniform}(a, b)$
 a, b : unknown

Q: What are the joint distributions of various order statistics and their marginal distributions? $\rightarrow 11/20$

➤ Definition. $X_1, \dots, X_n$ are called i.i.d. (independent, identically distributed) with cdf $\bar{F}$ /pdf f /pmf p if the random variables $X_1, \dots, X_n$ are independent and have a common marginal distribution with cdf F /pdf f /pmf p .

Intuition: (1) $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$
(2) $X_{(1)} = x, X_{(2)} \geq x \Rightarrow$ conditional dist of $X_{(2)} | X_{(1)} = x$

- Remark. In the discussion about order statistics, we only consider the case that $X_1, \dots, X_n$ are i.i.d.

change with x

exception
 $X_1 \in [0, 1]$
 $X_2 \in [2, 3]$
 $X_{(1)}, X_{(2)}$ indep.

⊙ Note. Although $X_1, \dots, X_n$ are independent, their order statistics $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ are not independent in general.