joint pmf → $P(\mathbf{1}_{A_1}=1, \mathbf{1}_{A_2}=0, \mathbf{1}_{A_3}=1)$ (0 or 1, 0 or 1, 0 or 1)

$$= P(A_1 \cap A_2^c \cap A_3) \overset{}{=} P(A_1)P(A_2^c)P(A_3)$$

by Thm in LNp.4-24

$$= P(\mathbf{1}_{A_1}=1)P(\mathbf{1}_{A_2}=0)P(\mathbf{1}_{A_3}=1).$$

marginal pmf

product of marginal pmfs

11/14

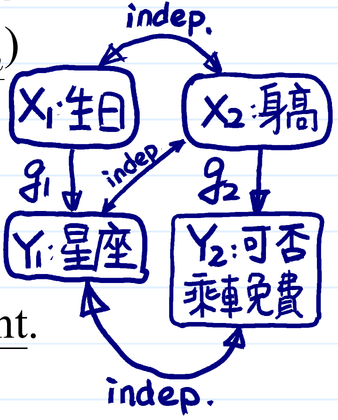
$$\begin{tikzcd} \Omega \arrow[d, "X"] \arrow[dd, bend right, red] \\ \mathbb{R}^n \arrow[d, "Y_i=g_i(X_i)"] \\ \mathbb{R}^n \end{tikzcd}$$

➢ **Theorem.** If $\mathbf{X}=(X_1, \ldots, X_n)$ are independent and

$$Y_i = g_i(X_i),\ i=1, \ldots, n,$$

then $g_i : \mathbb{R}^1 \to \mathbb{R}^1$

$Y_1, \ldots, Y_n$ are independent.



$X_1$ ... $X_n$

generalization

$1 = i_0 < i_1 < \cdots < i_k = n$

$Y_1 = g_1(X_1, \ldots, X_{i_1}),$

$Y_2 = g_2(X_{i_1+1}, \ldots, X_{i_2}),$

$\cdots$

$Y_k = g_k(X_{i_{k-1}+1}, \ldots, X_{i_k}).$

different independent r.v.'s

**Proof.**

Let $A_i(y) = \{x : g_i(x) \le y\}$, $i=1, \ldots, n$, then

joint cdf → $F_{\mathbf{Y}}(y_1, \ldots, y_n) = P(Y_1 \le y_1, \ldots, Y_n \le y_n)$

same subset of Ω $= P(X_1 \in A_1(y_1), \ldots, X_n \in A_n(y_n))$ ← a cross product set

∵ $X_1, \cdots, X_n$ indep. $= P(X_1 \in A_1(y_1)) \times \cdots \times P(X_n \in A_n(y_n))$

same subset of Ω $= P(Y_1 \le y_1) \times \cdots \times P(Y_n \le y_n)$

$= F_{Y_1}(y_1) \times \cdots \times F_{Y_n}(y_n).$ ← product of marginal cdfs

---

⊙ **Theorem.** $\mathbf{X}=(X_1, \ldots, X_n)$ are independent if and only if there exist univariate functions $g_i(x)$, $i=1, \ldots, n$, such that

$g_i : \mathbb{R}^1 \to \mathbb{R}^1$

also hold for joint cdf

cf. Factorization Thm (LNp.7-21) & definition (LNp.7-20)

(a) when $X_1, \ldots, X_n$ are discrete r.v.'s with joint pmf $p_{\mathbf{X}}$,

$$p_{\mathbf{X}}(x_1, \ldots, x_n) \propto g_1(x_1)\times\cdots\times g_n(x_n),\ -\infty<x_i<\infty,\ i=1,\ldots,n.$$

for any $\mathbf{x}=(x_1, \cdots, x_n) \in \mathbb{R}^n$

(b) when $X_1, \ldots, X_n$ are continuous r.v.'s with joint pdf $f_{\mathbf{X}}$,

$$(*)\ f_{\mathbf{X}}(x_1, \ldots, x_n) \propto g_1(x_1)\times\cdots\times g_n(x_n),\ -\infty<x_i<\infty,\ i=1,\ldots,n.$$

Sketch of proof for (b). → proof for (a): $\int \to \Sigma$ (exercise)

(⇒, trivial)

(⇐) $$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, x_2, \ldots, x_n)\, dx_2 \cdots dx_n$$

the rest part (without $g_1(x_1)$) is irrelevant to $x_1$

by (*) $$\propto \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \boxed{g_1(x_1)} g_2(x_2) \cdots g_n(x_n)\, dx_2 \cdots dx_n \propto g_1(x_1).$$

Similarly, $f_{X_2}(x_2) \propto g_2(x_2), \ldots, f_{X_n}(x_n) \propto g_n(x_n)$

$$\Rightarrow\ f_{X_1}(x_1)\cdots f_{X_n}(x_n) \propto g_1(x_1)\cdots g_n(x_n)$$

by (*) $$\Rightarrow\ f_{\mathbf{X}}(x_1, \ldots, x_n) \propto f_{X_1}(x_1)\cdots f_{X_n}(x_n)$$

$$\Rightarrow\ f_{\mathbf{X}}(x_1, \ldots, x_n) = c \cdot f_{X_1}(x_1)\cdots f_{X_n}(x_n)$$

for some constant $c$.

Because $$\underline{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}} f_{\mathbf{X}}(x_1, x_2, \ldots, x_n)\, \underline{dx_1\cdots dx_n} = 1, \text{ and}$$

$$\prod_{i=1}^{n}\int_{-\infty}^{\infty} f_{X_i}(x_i)dx_i = \underline{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}} f_{X_1}(x_1)\cdots f_{X_n}(x_n)\, \underline{dx_1\cdots dx_n} = 1, \quad \Rightarrow \underline{c = 1}.$$

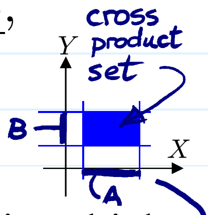
➢Example.

- If the joint pdf of (*X*, *Y*) is

$$f(x, y) \propto \underline{e^{-2x}}\,\underline{e^{-3y}}, \quad \underline{0 < x < \infty}, \underline{0 < y < \infty},$$

and $f(x, y)$=0, otherwise, i.e.,

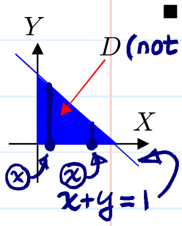
$$f(x, y) \propto \underbrace{e^{-2x}}_{g_1(x)}\overbrace{e^{-3y}}^{g_2(y)}\mathbf{1}_{(0,\infty)}(x)\overbrace{\mathbf{1}_{(0,\infty)}(y)}, \ \forall (x,y)\in\mathbb{R}^2$$



then *X* and *Y* are independent. Note that the region in which the joint pdf is nonzero can be expressed in the form

a cross-product set $\{(x, y): x\in A, y\in B\}$. $\equiv D \Rightarrow \mathbb{1}_D(x,y) = \mathbb{1}_A(x)\cdot\mathbb{1}_B(y)$

- Suppose that the joint pdf of (*X*, *Y*) is



$$f(x, y) \propto \underline{xy}, \quad \underline{0 < x < 1}, \underline{0 < y < 1}, \underline{0 < x + y < 1},$$

$g_1(x)$? $g_2(y)$?

and $f(x, y)$=0, otherwise, i.e., $f(x, y) \propto xy \cdot \mathbf{1}_D(x, y)$, $\forall (x,y)\in\mathbb{R}^2$

not a function of the form $\mathbb{1}_A(x)\mathbb{1}_B(y)$

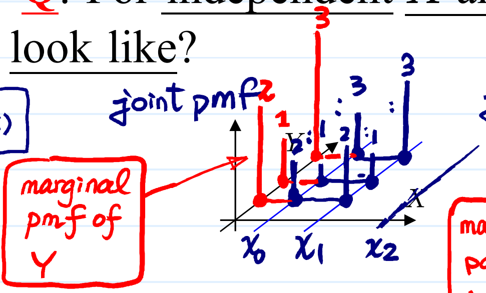
*X* and *Y* are not independent.

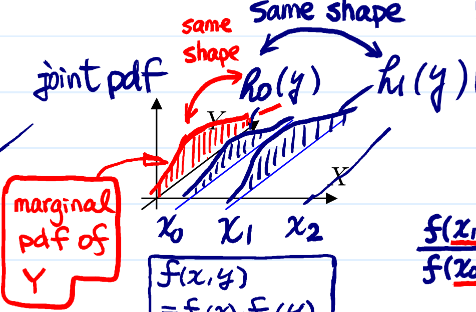
---

➢**Q**: For independent *X* and *Y*, how should their joint pdf/pmf look like?

$p(x,y) = p_X(x)p_Y(y)$

(2a+2b):
(1a+1b):
(3a+3b)
=2:1:3





$f(x,y) = f_X(x)\cdot f_Y(y)$

$\int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty}\frac{f_X(x)}{f_X(x_0)}h_0(y)dx$

depends on $x_0$ & $x_1$

$$\underline{\frac{h_1(y)}{h_0(y)}} = \underline{\text{a constant}} \quad \text{for all } y$$

$$\frac{f(x_1,y)}{f(x_0,y)} = \frac{f_X(x_1)\cancel{f_Y(y)}}{f_X(x_0)\cancel{f_Y(y)}} = \frac{f_X(x_1)}{f_X(x_0)}$$

❖ **Reading**: textbook, Sec 6.2

## Transformation

Recall Transformation for univariate r.v. (LNp.5-12 & 6-7~12)

- **Q**: Given the joint distribution of **X**=($X_1$, …,$X_n$), how to find the distribution of **Y**=($Y_1$, …,$Y_k$), where

$$\underline{Y_1}=\underline{g_1}(\underline{X_1, \ldots, X_n}) : \underline{\mathbb{R}^n}\to\underline{\mathbb{R}},$$

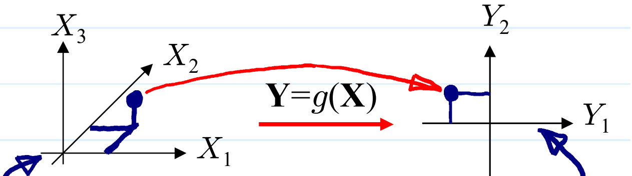
$$\ldots,$$

$$\underline{Y_k}=\underline{g_k}(\underline{X_1, \ldots, X_n}) : \underline{\mathbb{R}^n}\to\underline{\mathbb{R}},$$

denoted by $(g_1(\mathbf{X}), \cdots, g_k(\mathbf{X}))$

$$\underline{\mathbf{Y}}=\underline{g}(\underline{\mathbf{X}}),\ \underline{g}: \underline{\mathbb{R}^n}\to\underline{\mathbb{R}^k}.$$

n=3 k=2



Given
joint cdf $F_{\mathbf{X}}$
joint pmf $p_{\mathbf{X}}$
joint pdf $f_{\mathbf{X}}$

**Q**: What's the
joint cdf $F_{\mathbf{Y}}$
joint pmf $p_{\mathbf{Y}}$
joint pdf $f_{\mathbf{Y}}$

➢ The following methods are useful:

1. Method of Events (→ pmf)
2. Method of Cumulative Distribution Function
3. Method of Probability Density Function
4. Method of Moment Generating Function (chapter 7)

➢ Method of Events

original prob. space $\Omega, P$ → 1st prob. space ← original — new → 2nd prob. space

$\mathbb{R}^n$ (sample space) $P_X$ $\mathbb{R}^k$ $P_Y$

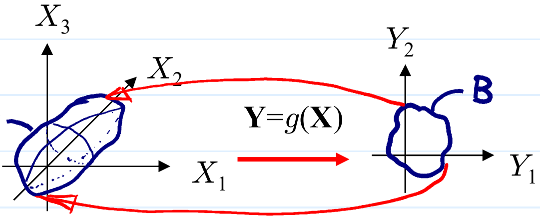
- Theorem. The distribution of **Y** is determined by the distribution of **X** as follows: for any event $B\subset\mathbb{R}^k$,

$$P_{\mathbf{Y}}(\mathbf{Y}\in B)=P_{\mathbf{X}}(\mathbf{X}\in A),$$

where $A=g^{-1}(B)\subset\mathbb{R}^n$.

not imply the inverse function of $g$ exists

same event in $\Omega$



- Example. Let **X** be a discrete random vector taking values

$$\mathbf{x}_i=(x_{1i}, x_{2i}, \dots, x_{ni}),\ i=1, 2, \dots,$$

range of **X**

(i.e., $\mathcal{X}=\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots\}$) with joint pmf $p_{\mathbf{X}}$.

Then, $\mathbf{Y}=g(\mathbf{X})$ is also a discrete random vector.

Y can take values in $\mathcal{Y}=g(\mathcal{X})=\{g(\mathbf{x}_1), g(\mathbf{x}_2), \cdots\}$

Suppose that **Y** takes values on $\mathbf{y}_j$, $j= 1, 2, \dots$. To determine the joint pmf of **Y**, by taking $B=\{\mathbf{y}_j\}$, we have

$$A=\{\mathbf{x}_i\in\mathcal{X}: g(\mathbf{x}_i)=\mathbf{y}_j\}$$

finite or countably infinite set

and hence, the joint pmf of **Y** is

by (d) in LNp.7-8

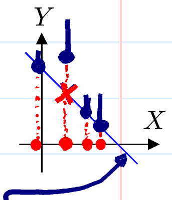
$$p_{\mathbf{Y}}(\mathbf{y}_j)=P_{\mathbf{Y}}(\{\mathbf{y}_j\})=P_{\mathbf{X}}(A)=\sum_{\mathbf{x}_i\in A}p_{\mathbf{X}}(\mathbf{x}_i).$$

$(X,Y)\to Z$

$\mathbb{R}^2\to\mathbb{R}^1$

- Example. Let $X$ and $Y$ be random variables with the joint pmf $p(x, y)$. Find the distribution of $Z=X+Y$.

X & Y are discrete r.v.'s



$x+y=z$ fixed

$\Rightarrow y=z-x$

□ $\{Z=z\}=\{(X, Y)\in\{(x, y): x+y=z\}\}$ B A

$$p_Z(z)=P_Z(\{z\})=P(X+Y=z)=\sum_{x\in\mathcal{X}_X}p(x, z-x)$$ (□)

range of X

□ When $X$ and $Y$ are independent,

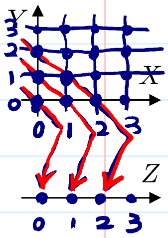
$$p(x, y)=p_X(x)p_Y(y),$$

So,

$$p_Z(z)=\sum_{x\in\mathcal{X}_X}p_X(x)p_Y(z-x).$$

which is referred to as the *convolution* of $p_X$ and $p_Y$.

□ (Exercise) $Z=X-Y$ Ans. $\sum_{y\in\mathcal{X}_Y}p(z+y, y)$



- **Theorem.** If $X$ and $Y$ are independent, and

$$X \sim \text{Poisson}(\lambda_1), \quad Y \sim \text{Poisson}(\lambda_2),$$

then $Z = X + Y \sim \text{Poisson}(\lambda_1+\lambda_2)$.

Proof. For $z$=0, 1, 2, …, the pmf $p_Z(z)$ of $Z$ is

all possible values of Z

pmf of Poisson($\lambda_1+\lambda_2$)

$$p_Z(z) = \sum_{x=0}^{z} p_X(x)p_Y(z-x) = \sum_{x=0}^{z} \frac{\boxed{e^{-\lambda_1}}\lambda_1^x}{x!}\frac{\boxed{e^{-\lambda_2}}\lambda_2^{z-x}}{(z-x)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{z!}\left(\sum_{x=0}^{z}\underbrace{\frac{z!}{x!(z-x)!}}_{\binom{z}{x}}\lambda_1^x\lambda_2^{z-x}\right) = \frac{e^{-(\lambda_1+\lambda_2)}}{z!}(\lambda_1+\lambda_2)^z.$$

by Binomial Thm (LNp 5-23)

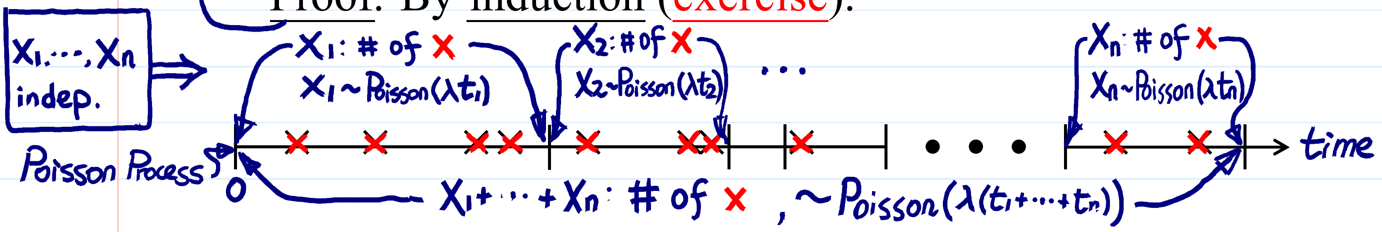
$Y_k = X_1+\cdots+X_k \sim$ Poisson($\lambda_1+\cdots+\lambda_k$)
$X_{k+1} \sim$ Poisson($\lambda_{k+1}$)
Are $Y_k$ & $X_{k+1}$ independent?
Ans. Yes, by the generalization in LNp.7-24

□ Corollary. If $X_1, \ldots, X_n$ are independent, and

$$X_i \sim \text{Poisson}(\lambda_i),\ i=1, \ldots, n,$$

then $X_1+\cdots+X_n \sim \text{Poisson}(\lambda_1+\cdots+\lambda_n)$.

Proof. By induction (exercise).



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## ➢ Method of cumulative distribution function

a special case of the method of events

replace the B by $(-\infty,y_1]\times\cdots\times(-\infty,y_n]$

1. In the $(X_1, \ldots, X_n)$ space, find the region that corresponds to $\{Y_1 \le y_1, \ldots, Y_k \le y_k\}$.

   $\mathbf{Y}=g(\mathbf{X})$, $\mathbf{Y}\in\mathbb{R}^k$, $\mathbf{X}\in\mathbb{R}^n$; $B(\subset\mathbb{R}^k) \to$ ; $A(\subset\mathbb{R}^n)$

2. Find $F_{\mathbf{Y}}(y_1, \ldots, y_k)=P(Y_1\le y_1, \ldots, Y_k \le y_k)$ by summing the joint pmf or integrating the joint pdf of $X_1, \ldots, X_n$ over the region identified in 1. Ⓐ

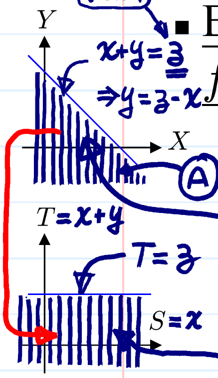
3. (for continuous case) Find the joint pdf of $\mathbf{Y}$ by differentiating $F_{\mathbf{Y}}(y_1, \ldots, y_k)$, i.e.,

$$f_{\mathbf{Y}}(y_1, \ldots, y_k) = \frac{\partial^k}{\partial y_1\cdots\partial y_k}F_{\mathbf{Y}}(y_1, \ldots, y_k).$$

X & Y are continuous r.v.'s

Fixed

- Example. $X$ and $Y$ are random variables with joint pdf $f(x, y)$. Find the distribution of $Z=X+Y$.



□ $\{Z\le z\} = \{(X, Y) \in \{(x, y): x+y\le z\}\}$. So, Ⓑ Ⓐ

$$F_Z(z) = P(Z \le z) = P(X+Y \le z)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)\,dydx$$
$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} f(s,t-s)\,dsdt \qquad \left(\text{set}\ \begin{cases} x = s \\ y = t-s \end{cases}\right)$$

$\begin{cases} s = x \\ t = y+x \end{cases}$

$J = \begin{vmatrix} dx/ds & dx/dt \\ dy/ds & dy/dt \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

$\Rightarrow dxdy = |J|dsdt = dsdt$

and $$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx \xleftrightarrow{\text{c.f.}} (\square)$$

in LNp.7-29 for discrete case.

$\Sigma \leftrightarrow \int$

$p \leftrightarrow f$

- ☐ When $X$ and $Y$ are independent,

$$f(x, y) = f_X(x)f_Y(y).$$

So, $$F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_X(x)f_Y(y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z-x} f_Y(y)\,dy\right] f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} F_Y(z-x)f_X(x)\,dx$$

$\boxed{\int_{-\infty}^{\infty} F_X(z-y)f_Y(y)dy}$

Integration by parts (exercise)

$\boxed{\frac{d}{dx}F_X(x)}$

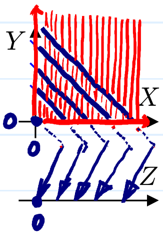
$\boxed{Z=X+Y=Y+X}$

which is referred to as the *convolution* of $F_X$ and $F_Y$, and

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,dx \xleftrightarrow{\text{cf.}} \boxed{\text{convolution of pmfs (LNp.7-29)}}$$

which is referred to as the *convolution* of $f_X$ and $f_Y$.

- ☐ (exercise) $Z=X-Y$. → Ans. $f_Z(z)=\int_{-\infty}^{\infty} f_{X,Y}(z+y, y)dy$



- ■ Theorem. If $X$ and $Y$ are independent, and

$$X \sim \text{Gamma}(\alpha_1, \lambda),\quad Y \sim \text{Gamma}(\alpha_2, \lambda),$$

then

$$Z = X + Y \sim \text{Gamma}(\alpha_1+\alpha_2, \lambda). \leftarrow \boxed{\text{intuition}}$$

Proof. For $z \geq 0$,

$$f_Z(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^z \overset{z^{\alpha_1-1}\left(\frac{x}{z}\right)^{\alpha_1-1}}{x^{\alpha_1-1}}\overset{z^{\alpha_2-1}\left(1-\frac{x}{z}\right)^{\alpha_2-1}}{(z-x)^{\alpha_2-1}}e^{-\overset{\lambda(x+(z-x))}{\boxed{\lambda z}}}\,dx$$

$\boxed{\begin{array}{l}\text{Let } y=x/z\\ \Rightarrow x=zy\\ \frac{dx}{dy}=z\\ \Rightarrow dx=z\,dy\end{array}}$

$$= \frac{\lambda^{\alpha_1+\alpha_2}e^{-\lambda z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 z^{(\alpha_1-1)+(\alpha_2\cancel{-1})\cancel{+1}}\underbrace{y^{\alpha_1-1}(1-y)^{\alpha_2-1}}_{\text{Beta function (LNp.6-27)}}dy$$

$$= \underbrace{\frac{\lambda^{\alpha_1+\alpha_2}z^{(\alpha_1+\alpha_2)-1}e^{-\lambda z}}{\cancel{\Gamma(\alpha_1)\Gamma(\alpha_2)}}\times\frac{\cancel{\Gamma(\alpha_1)\Gamma(\alpha_2)}}{\Gamma(\alpha_1+\alpha_2)}}_{\text{pdf of Gamma}(\alpha_1+\alpha_2,\lambda)}.$$

and $f_Z(z) = 0$, for $z < 0$.

- ☐ Corollary. If $X_1, \ldots, X_n$ are independent, and

$$X_i \sim \text{Gamma}(\alpha_i, \lambda),\ i=1, \ldots, n,$$

then $X_1 + \cdots + X_n \sim \text{Gamma}(\alpha_1+\cdots+\alpha_n, \lambda)$. ← $\boxed{\text{intuition}}$

Proof. By induction (exercise).

- ☐ Corollary. If $X_1, \ldots, X_n$ are independent, and

$$X_i \sim \underbrace{\text{Exponential}(\lambda)}_{\rightarrow\ \text{Gamma}(1,\lambda)},\ i=1, \ldots, n,$$

then $X_1 + \cdots + X_n \sim \text{Gamma}(n, \lambda)$. ← $\boxed{\text{intuition}}$

Proof. (exercise).

check Textbook Sec.6.3.3

■ **Theorem.** If $X_1, \ldots, X_n$ are independent, and $X_i \sim \text{Normal}(\mu_i, \sigma_i^2)$, $i=1, \ldots, n$, then $X_1 + \cdots + X_n \sim \text{Normal}(\mu_1+\cdots+\mu_n, \sigma_1^2+\cdots+\sigma_n^2)$.

$E(X_1+\cdots+X_n) = E(X_1)+\cdots+E(X_n)$

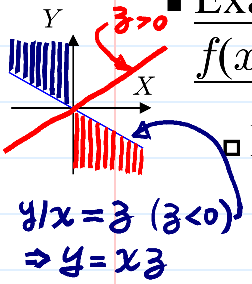
$Var(X_1+\cdots+X_n) = Var(X_1)+\cdots+Var(X_n)$

(Chapter 7)

**Proof.** (exercise).

■ **Example.** $X$ and $Y$ are random variables with joint pdf $f(x, y)$. Find the distribution of $Z=Y/X$.

X & Y: continuous r.v.'s



☐ Let $Q_z = \{(x,y) : y/x \le z\}$

$\quad\; Ⓐ = \{(x,y) : x<0, y \ge zx\}$

$\qquad\;\cup \{(x,y) : x>0, y \le zx\}$

B: {Z≤z}

P(Z≤z)

then, $F_Z(z) = \int\int_{Q_z} f(x,y)\, dxdy$

$$= \int_{-\infty}^{0}\int_{xz}^{\infty} + \int_{0}^{\infty}\int_{-\infty}^{xz} f(x,y)\, dydx$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{z} + \int_{0}^{\infty}\int_{-\infty}^{z} f(s,st)\,\boxed{|s|}\,dtds$$

|J|

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z} |s| f(s,st)\, dtds$$

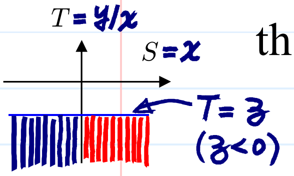
$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} |s| f(s,st)\, dsdt$$

$\left(\text{set } \begin{cases} x = s \\ y = st \end{cases}\right)$

$s = x,\ t = y/x$

$J = \begin{vmatrix} \frac{dx}{ds} & \frac{dx}{dt} \\ \frac{dy}{ds} & \frac{dy}{dt} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ t & s \end{vmatrix} = s$

$dxdy = |s|\,dsdt$



and, $f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} |x| f(x, zx)\, dx$

分母 分子

☐ When $X$ and $Y$ are independent,

$$f(x, y) = f_X(x) f_Y(y).$$

So, $F_Z(z) = \int_{-\infty}^{z}\int_{-\infty}^{\infty} |s| f_X(s) f_Y(st)\, dsdt$ — (*)

and, $f_Z(z) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(zx)\, dx$

分母 分子

Hint.
$Z = XY$
$= Y/(X^{-1})$
Find the joint pdf of $(X^{-1}, Y)$

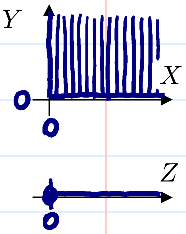
☐ (exercise) $Z=XY$

☐ If $X$ and $Y$ are independent,

$$X \sim \text{exponential}(\lambda_1), \text{ and } Y \sim \text{exponential}(\lambda_2),$$

e.g. ratio of 2 lifetimes

Let $Z=Y/X$. The pdf of $Z$ is



$$f_Z(z) = \int_0^\infty x\left(\lambda_1 e^{-\lambda_1 x}\right)\left[\lambda_2 e^{-\lambda_2(xz)}\right]\, dx$$

$$= \frac{\lambda_1\lambda_2\Gamma(2)}{(\lambda_1+\lambda_2 z)^2}\int_0^\infty \underbrace{\frac{(\lambda_1+\lambda_2 z)^2}{\Gamma(2)} x^{2-1} e^{-(\lambda_1+\lambda_2 z)x}}_{\text{pdf of Gamma}(2,\,(\lambda_1+\lambda_2)z)}\, dx$$

$$= \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2 z)^2}$$

for $z \ge 0$, and $0$ for $z < 0$.

a special case of the method of cdf (see proof of the Thm)

And, the cdf of $Z$ is

$$F_Z(z) = \int_0^z f_Z(t)\,dt = \int_0^z \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2 t)^2}\,dt$$

$$= -\frac{\lambda_1\lambda_2}{\lambda_2}(\lambda_1+\lambda_2 t)^{-1}\Big|_0^z = 1-\frac{\lambda_1}{\lambda_1+\lambda_2 z} = \begin{cases}1, & z\uparrow\infty\\ 0, & z\downarrow 0\end{cases}$$

for $z \geq 0$, and 0 for $z<0$.

## ➢ Method of probability density function

- **Theorem.** Let $\mathbf{X}=(X_1, \ldots, X_n)$ be continuous random variables with the joint pdf $f_{\mathbf{X}}(x_1, \ldots, x_n)$. Let

$$\mathbf{Y}=(Y_1, \ldots, Y_n)= g(\mathbf{X}),$$

$(g_1, g_2, \cdots, g_n)$, $g_i: \mathbb{R}^n \to \mathbb{R}^1$, $Y_i = g_i(X_1, \cdots, X_n)$

where ① $g$ is 1-to-1, so that its inverse exists and is denoted by

$$\mathbf{x}=g^{-1}(\mathbf{y}) = w(\mathbf{y}) = (w_1(\mathbf{y}), w_2(\mathbf{y}), \ldots, w_n(\mathbf{y})).$$

$w_i: \mathbb{R}^n \to \mathbb{R}^1$, $X_i = w_i(Y_1, \cdots, Y_n)$

Assume ② $w$ have continuous partial derivatives. Let

Jacobin

determinant

$$J(y_1, \cdots, y_n) = J = \begin{vmatrix} \frac{\partial w_1(\mathbf{y})}{\partial y_1} & \frac{\partial w_1(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_1(\mathbf{y})}{\partial y_n} \\ \frac{\partial w_2(\mathbf{y})}{\partial y_1} & \frac{\partial w_2(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_2(\mathbf{y})}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial w_n(\mathbf{y})}{\partial y_1} & \frac{\partial w_n(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_n(\mathbf{y})}{\partial y_n} \end{vmatrix}_{n\times n} = \left|\left[\frac{\partial w_i}{\partial y_j}\right]\right|$$

Note. It becomes $J^{-1}$ if use $\left|\left[\partial g_i/\partial x_j\right]\right|$

Then $\quad f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(g^{-1}(\mathbf{y})) \times |J|,$ c.f. Thm in LNp.6-10~11

absolute value of J

for $\mathbf{y}$ s.t. $\mathbf{y}=g(\mathbf{x})$ for some $\mathbf{x}$ and $f_{\mathbf{Y}}(\mathbf{y})=0$, otherwise.

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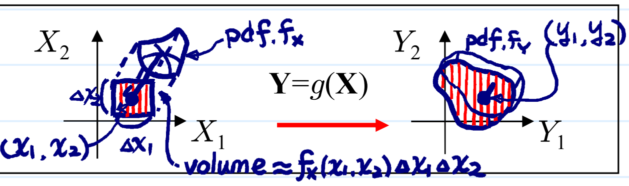
Recall. The question in LNp.6-11

(**Q**: What is the role of $|J|$?)



Proof. $F_{\mathbf{Y}}(y_1, \ldots, y_n) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} f_{\mathbf{Y}}(t_1, \ldots, t_n)\, dt_n \cdots dt_1$

$$= \int \cdots \int_{\substack{(x_1,\ldots,x_n):\\ Y_1 = g_1(x_1,\ldots,x_n)\le y_1\\ \cdots\\ Y_n = g_n(x_1,\ldots,x_n)\le y_n}} f_{\mathbf{X}}(x_1, \ldots, x_n)\, dx_n \cdots dx_1.$$

method of cdf check 1~3 in LNp.7-31

It then follows from an exercise in advanced calculus that

$$f_{\mathbf{Y}}(y_1, \ldots, y_n) = \frac{\partial^n}{\partial y_1 \cdots \partial y_n} F_{\mathbf{Y}}(y_1, \ldots, y_n)$$

$$= f_{\mathbf{X}}(\underbrace{w_1(\mathbf{y}), \ldots, w_n(\mathbf{y})}_{g^{-1}(\mathbf{y})}) \times |J|.$$

Functions of $\mathbf{X}$

□ **Remark.** When the dimensionality of $\mathbf{Y}$ (denoted by $k$) is less than $n$, we can choose another $n-k$ transformations $\mathbf{Z}$ such that

$$(\overset{k\text{-dim}}{\mathbf{Y}}, \overset{(n-k)\text{-dim}}{\mathbf{Z}}) = g(\mathbf{X})$$

n-dim

$\mathbb{R}^n \to \mathbb{R}^n$, $g^{-1}$ exists, $g^{-1}$: differentiable

satisfy the assumptions in above theorem.