11/13

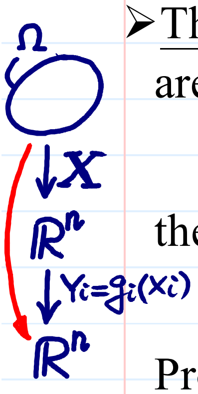
joint pmf → $P(\underbrace{\mathbf{1}_{A_1} = 1}, \underbrace{\mathbf{1}_{A_2} = 0}, \underbrace{\mathbf{1}_{A_3} = 1})$ (0 or 1, 0 or 1, 0 or 1)

$$= P(A_1 \cap A_2^c \cap A_3) \overset{\text{by Thm in LNp.4-24}}{=} P(A_1)P(A_2^c)P(A_3)$$

$$= \underline{P(\mathbf{1}_{A_1} = 1)}\,\underline{P(\mathbf{1}_{A_2} = 0)}\,\underline{P(\mathbf{1}_{A_3} = 1)}.$$

marginal pmf

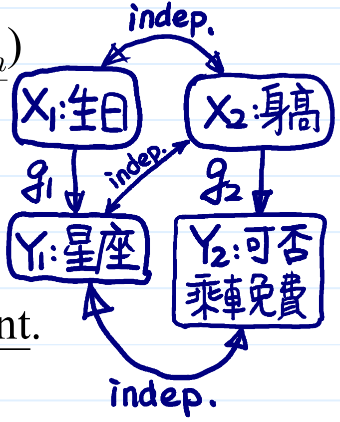
product of marginal pmfs



➢ **Theorem.** If $\mathbf{X}=(X_1, \ldots, X_n)$ are independent and

$$Y_i = g_i(X_i),\ i=1, \ldots, n,$$

then $Y_1, \ldots, Y_n$ are independent. ($g_i : \mathbb{R}^1 \to \mathbb{R}^1$)



generalization

$$1 = i_0 < i_1 < \cdots < i_k = n$$ (X_1, X_n)

$$\begin{aligned} Y_1 &= g_1(X_1, \ldots, X_{i_1}), \\ Y_2 &= g_2(X_{i_1+1}, \ldots, X_{i_2}), \\ &\cdots \\ Y_k &= g_k(X_{i_{k-1}+1}, \ldots, X_{i_k}). \end{aligned}$$

different independent r.v.'s

Proof.

Let $A_i(y) = \{x : g_i(x) \le y\}$, $i=1, \ldots, n$, then

$$\begin{aligned} \text{joint cdf} \to F_{\mathbf{Y}}(y_1, \ldots, y_n) &= P(Y_1 \le y_1, \ldots, Y_n \le y_n) \\ &\overset{\text{same subset of }\Omega}{=} P(X_1 \in A_1(y_1), \ldots, X_n \in A_n(y_n)) \quad \text{a cross product set} \\ &\overset{\because X_1,\ldots,X_n \text{ indep.}}{=} P(X_1 \in A_1(y_1)) \times \cdots \times P(X_n \in A_n(y_n)) \\ &\overset{\text{same subset of }\Omega}{=} P(Y_1 \le y_1) \times \cdots \times P(Y_n \le y_n) \\ &= F_{Y_1}(y_1) \times \cdots \times F_{Y_n}(y_n). \end{aligned}$$

product of marginal cdfs

Theorem. $\mathbf{X}=(X_1, \ldots, X_n)$ are independent if and only if there exist univariate functions $g_i(x)$, $i=1, \ldots, n$, such that ($g_i : \mathbb{R}^1 \to \mathbb{R}^1$)

also hold for joint cdf

cf. Factorization Thm (LNp.7-21) & definition (LNp.7-20)

(a) when $X_1, \ldots, X_n$ are discrete r.v.'s with joint pmf $p_{\mathbf{X}}$,

$$p_{\mathbf{X}}(x_1, \ldots, x_n) \propto g_1(x_1)\times\cdots\times g_n(x_n),\ -\infty<x_i<\infty,\ i=1,\ldots,n.$$

for any $\underset{\sim}{x}=(x_1,\cdots,x_n) \in \mathbb{R}^n$

(b) when $X_1, \ldots, X_n$ are continuous r.v.'s with joint pdf $f_{\mathbf{X}}$,

(*) — $$f_{\mathbf{X}}(x_1, \ldots, x_n) \propto g_1(x_1)\times\cdots\times g_n(x_n),\ -\infty<x_i<\infty,\ i=1,\ldots,n.$$

Sketch of proof for (b). → proof for (a): $\int \to \Sigma$ (exercise)

($\Rightarrow$, trivial)

($\Leftarrow$) $$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, x_2, \ldots, x_n)\, dx_2 \cdots dx_n$$

$$\overset{\text{by }(*)}{\propto} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \boxed{g_1(x_1)}\, g_2(x_2) \cdots g_n(x_n)\, dx_2 \cdots dx_n \propto g_1(x_1).$$

the rest part (without $g_1(x_1)$) is irrelevant to x_1

Similarly, $f_{X_2}(x_2) \propto g_2(x_2), \ldots, f_{X_n}(x_n) \propto g_n(x_n)$

$$\begin{aligned} &\Rightarrow f_{X_1}(x_1) \cdots f_{X_n}(x_n) \propto g_1(x_1) \cdots g_n(x_n) \\ &\overset{\text{by }(*)}{\Rightarrow} f_{\mathbf{X}}(x_1, \ldots, x_n) \propto f_{X_1}(x_1) \cdots f_{X_n}(x_n) \\ &\Rightarrow f_{\mathbf{X}}(x_1, \ldots, x_n) = c \cdot f_{X_1}(x_1) \cdots f_{X_n}(x_n) \end{aligned}$$

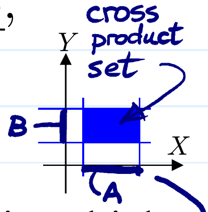
for some constant c.

Because $$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1,x_2,\ldots,x_n)\,dx_1\cdots dx_n = 1, \text{ and}$$

$$\prod_{i=1}^{n}\int_{-\infty}^{\infty} f_{X_i}(x_i)dx_i = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{X_1}(x_1)\cdots f_{X_n}(x_n)\,dx_1\cdots dx_n = 1, \quad\Rightarrow c = 1.$$

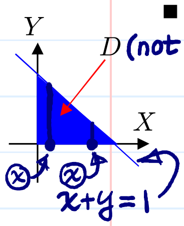
➢ Example.

- If the joint pdf of (X, Y) is
  $$f(x,y) \propto e^{-2x}e^{-3y},\quad 0<x<\infty,\ 0<y<\infty,$$
  and $f(x, y)$=0, otherwise, i.e.,
  $$f(x,y) \propto \underbrace{e^{-2x}}_{g_1(x)}\overbrace{e^{-3y}\mathbf{1}_{(0,\infty)}(x)\mathbf{1}_{(0,\infty)}(y)}^{g_2(y)},\ \forall (x,y)\in\mathbb{R}^2$$



  then X and Y are independent. Note that the region in which the joint pdf is nonzero can be expressed in the form

  a cross-product set $\{(x, y): x\in A, y\in B\} \equiv D \Rightarrow \mathbb{1}_D(x,y)=\mathbb{1}_A(x)\cdot\mathbb{1}_B(y)$



- Suppose that the joint pdf of (X, Y) is
  $$f(x,y) \propto \underset{g_1(x)?}{x}\underset{g_2(y)?}{y},\quad 0<x<1,\ 0<y<1,\ 0<x+y<1,$$
  and $f(x, y)$=0, otherwise, i.e., $f(x,y) \propto xy\cdot\mathbf{1}_D(x,y), \forall (x,y)\in\mathbb{R}^2$

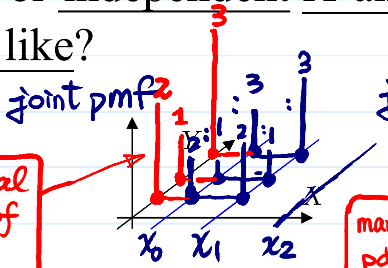
  not a function of the form $\mathbb{1}_A(x)\mathbb{1}_B(y)$

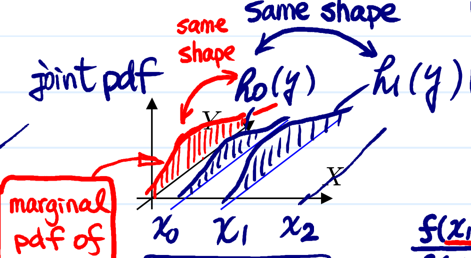
  X and Y are not independent.

➢ **Q**: For independent X and Y, how should their joint pdf/pmf look like?

$P(x,y) = P_X(x)P_Y(y)$

$(2a+2b)$:
$(1a+1b)$:
$(3a+3b)$
$=2:1:3$





$f(x,y) = f_X(x)\cdot f_Y(y)$

$\int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty}\frac{f_X(x)}{f_X(x_0)}h_0(y)dx$

depends on x_0 & x_1

$$\frac{h_1(y)}{h_0(y)} = \text{a constant for all } y$$

$$\frac{f(x_1,y)}{f(x_0,y)} = \frac{f_X(x_1)\cancel{f_Y(y)}}{f_X(x_0)\cancel{f_Y(y)}} = \frac{f_X(x_1)}{f_X(x_0)}$$

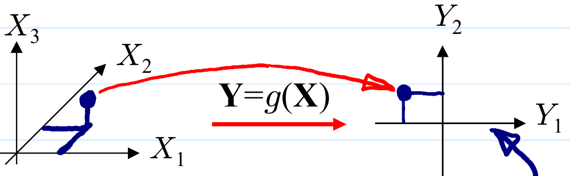
❖ **Reading**: textbook, Sec 6.2

Transformation

Recall. Transformation for univariate r.v. (LNp.5-12 & 6-7~12)

- **Q**: Given the joint distribution of $\mathbf{X}=(X_1, \ldots, X_n)$, how to find the distribution of $\mathbf{Y}=(Y_1, \ldots, Y_k)$, where
  $$Y_1=g_1(X_1,\ldots,X_n) : \mathbb{R}^n\to\mathbb{R},$$
  $$\ldots,$$
  $$Y_k=g_k(X_1,\ldots,X_n) : \mathbb{R}^n\to\mathbb{R},$$
  denoted by $\overset{(g_1(\mathbf{X}),\cdots,g_k(\mathbf{X}))}{\mathbf{Y}=\mathbf{g}(\mathbf{X})}$, $g:\mathbb{R}^n\to\mathbb{R}^k$.

n=3 k=2.



Given
joint cdf $F_{\mathbf{X}}$ → joint cdf $F_{\mathbf{Y}}$
joint pmf $p_{\mathbf{X}}$ → joint pmf $p_{\mathbf{Y}}$
joint pdf $f_{\mathbf{X}}$ → joint pdf $f_{\mathbf{Y}}$

Q: What's the joint cdf $F_{\mathbf{Y}}$, joint pmf $p_{\mathbf{Y}}$, joint pdf $f_{\mathbf{Y}}$

➢The following methods are useful:

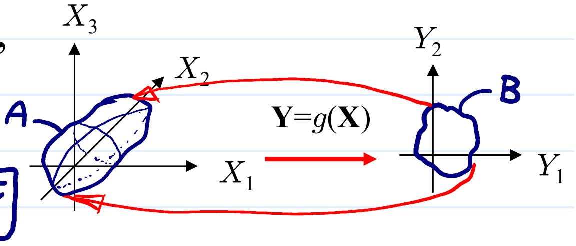
1. Method of Events (→ pmf)
2. Method of Cumulative Distribution Function
3. Method of Probability Density Function
4. Method of Moment Generating Function (chapter 7)

➢Method of Events

original prob. space Ω, P → 1st prob. space $\overset{\text{original}}{\longleftrightarrow}$ $\overset{\text{new}}{}$ 2nd prob. space

$\mathbb{R}^n$ (sample space) P_X $\quad$ $\mathbb{R}^k$ P_Y

- Theorem. The distribution of $\mathbf{Y}$ is determined by the distribution of $\mathbf{X}$ as follows: for any event $B \subset \mathbb{R}^k$,



$$P_{\mathbf{Y}}(\mathbf{Y} \in B) = P_{\mathbf{X}}(\mathbf{X} \in A),$$

where $A = g^{-1}(B) \subset \mathbb{R}^n$.

Same event in Ω

not imply the inverse function of g exists

- Example. Let $\mathbf{X}$ be a discrete random vector taking values

range of $\mathbf{X}$

$$\mathbf{x}_i = (x_{1i}, x_{2i}, \ldots, x_{ni}),\ i=1, 2, \ldots,$$

(i.e., $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$) with joint pmf $p_{\mathbf{X}}$.

Then, $\mathbf{Y}=g(\mathbf{X})$ is also a discrete random vector.

$\mathbf{Y}$ can take values in $\mathcal{Y} = g(\mathcal{X}) = \{g(\mathbf{x}_1), g(\mathbf{x}_2), \cdots\}$

Suppose that $\mathbf{Y}$ takes values on $\mathbf{y}_j$, $j=1, 2, \ldots$. To determine the joint pmf of $\mathbf{Y}$, by taking $B=\{\mathbf{y}_j\}$, we have

$$A = \{\mathbf{x}_i \in \mathcal{X} : g(\mathbf{x}_i) = \mathbf{y}_j\}$$

finite or countably infinite set

and hence, the joint pmf of $\mathbf{Y}$ is

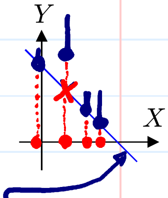
by (d) in LNp.7-8

$$p_{\mathbf{Y}}(\mathbf{y}_j) = P_{\mathbf{Y}}(\{\mathbf{y}_j\}) = P_{\mathbf{X}}(A) = \sum_{\mathbf{x}_i \in A} p_{\mathbf{X}}(\mathbf{x}_i).$$

$(X,Y) \to Z$
$\mathbb{R}^2 \to \mathbb{R}^1$

- Example. Let X and Y be random variables with the joint pmf $p(x, y)$. Find the distribution of $Z=X+Y$.

X & Y are discrete r.v.'s



$x+y=z$ fixed
$\Rightarrow y = z-x$

  - ▫ $\{Z=z\} = \{(X, Y) \in \{(x, y): x+y=z\}\}$ (B) (A)

$$p_Z(z) = P_Z(\{z\}) = P(X+Y=z) = \sum_{x \in \mathcal{X}_X} p(x, z-x)$$

(□) range of X

  - ▫ When X and Y are ~~independent~~,

$$p(x, y) = p_X(x)p_Y(y),$$

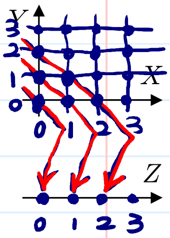
So,

$$p_Z(z) = \sum_{x \in \mathcal{X}_X} p_X(x) p_Y(z-x).$$

which is referred to as the *convolution* of p_X and p_Y.

independent

  - ▫ (Exercise) $Z=X-Y=X+(-Y)$ $\quad$ Ans. $\sum_{y \in \mathcal{X}_Y} p(z+y, y)$



- **Theorem**. If X and Y are independent, and

$$X \sim \text{Poisson}(\lambda_1), \quad Y \sim \text{Poisson}(\lambda_2),$$

then $Z = X + Y \sim \text{Poisson}(\lambda_1+\lambda_2)$.

Proof. For z=0, 1, 2, …, the pmf $p_Z(z)$ of Z is

all possible values of Z

pmf of Poisson($\lambda_1+\lambda_2$)

$$p_Z(z) = \sum_{x=0}^{z} p_X(x)p_Y(z-x) = \sum_{x=0}^{z} \frac{\boxed{e^{-\lambda_1}}\lambda_1^x}{x!}\frac{\boxed{e^{-\lambda_2}}\lambda_2^{z-x}}{(z-x)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{z!}\left(\sum_{x=0}^{z}\overset{\binom{z}{x}\,=}{\frac{z!}{x!(z-x)!}}\lambda_1^x\lambda_2^{z-x}\right) = \frac{e^{-(\lambda_1+\lambda_2)}}{z!}(\lambda_1+\lambda_2)^z.$$

by Binomial Thm (LN p.5-23)

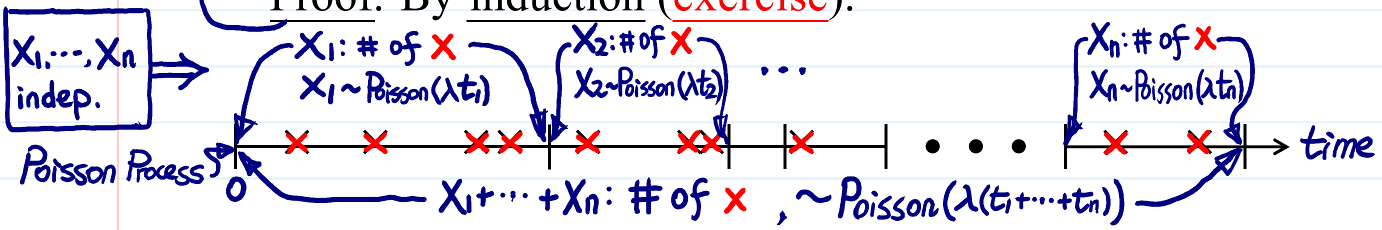
$Y_k = X_1 + \cdots + X_k \sim \text{Poisson}(\lambda_1 + \cdots + \lambda_k)$
$X_{k+1} \sim \text{Poisson}(\lambda_{k+1})$
Are Y_k & X_{k+1} independent?
Ans. Yes, by the generalization in LN p.7-24

□ **Corollary**. If $X_1, \ldots, X_n$ are independent, and

$$X_i \sim \text{Poisson}(\lambda_i),\ i=1, \ldots, n,$$

then $X_1+\cdots+X_n \sim \text{Poisson}(\lambda_1+\cdots+\lambda_n)$.

Proof. By induction (exercise).



Method of cumulative distribution function

a special case of the method of events

replace the B by $(-\infty, y_1] \times \cdots \times (-\infty, y_n]$

1. In the $(X_1, \ldots, X_n)$ space, find the region that corresponds to

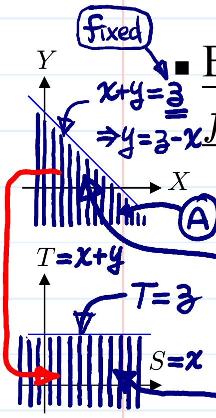
$\mathbb{Y} = g(\mathbb{X})$ ($\mathbb{Y} \in \mathbb{R}^k$, $\mathbb{X} \in \mathbb{R}^n$) $\boxed{B(\subset \mathbb{R}^k)} \rightarrow \{Y_1 \le y_1, \ldots, Y_k \le y_k\}$. $\boxed{A(\subset \mathbb{R}^n)}$

2. Find $F_{\mathbf{Y}}(y_1, \ldots, y_k) = P(Y_1 \le y_1, \ldots, Y_k \le y_k)$ by summing the joint pmf or integrating the joint pdf of $X_1, \ldots, X_n$ over the region identified in 1. Ⓐ

3. (for continuous case) Find the joint pdf of $\mathbf{Y}$ by differentiating $F_{\mathbf{Y}}(y_1, \ldots, y_k)$, i.e.,

$$f_{\mathbf{Y}}(y_1, \ldots, y_k) = \frac{\partial^k}{\partial y_1 \cdots \partial y_k} F_{\mathbf{Y}}(y_1, \ldots, y_k).$$

X & Y are continuous r.v.'s



- **Example**. X and Y are random variables with joint pdf $f(x, y)$. Find the distribution of $Z=X+Y$.

□ $\{Z \le z\} = \{(X, Y) \in \{(x, y): x+y \le z\}\}$. So,

Ⓑ Ⓐ

$$F_Z(z) = P(Z \le z) = P(X + Y \le z)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)\, dydx$$

$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} f(s, t-s)\, dsdt \qquad \left(\text{set } \begin{cases} x = s \\ y = t-s \end{cases}\right)$$

$\begin{cases} s = x \\ t = y+x \end{cases}$

$J = \begin{vmatrix} dx/ds & dx/dt \\ dy/ds & dy/dt \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

$\Rightarrow dxdy = |J|dsdt = dsdt$

and $f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f(x, z-x)dx \xleftrightarrow{\text{c.f.}} (\square)$ in LNp.7-29 for discrete case. $\Sigma \leftrightarrow \int$, $p \leftrightarrow f$

- ▫ When X and Y are independent,

$$f(x, y) = f_X(x)f_Y(y).$$

So, $F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_X(x)f_Y(y)\, dydx$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z-x} f_Y(y)\, dy\right] f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} F_Y(z-x)f_X(x)\, dx$$

$\int_{-\infty}^{\infty}F_X(z-y)f_Y(y)dy$

Integration by parts & $y=z-x \Rightarrow x=z-y$ (exercise)

$\frac{d}{dx}F_X(x)$

$Z=X+Y=Y+X$

which is referred to as the *convolution* of F_X and F_Y, and

independent

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\, dx$$

c.f.

convolution of pmfs (LNp.7-29)

which is referred to as the *convolution* of f_X and f_Y.

independent

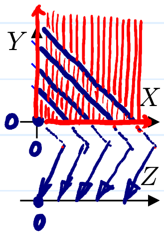
- ▫ (exercise) $Z=X-Y$. → Ans. $f_Z(z)=\int_{-\infty}^{\infty}f_{X,Y}(z+y, y)dy$

- ▪ Theorem. If X and Y are independent, and

$$X\sim \text{Gamma}(\alpha_1, \lambda),\quad Y\sim \text{Gamma}(\alpha_2, \lambda),$$

then

$$Z = X + Y \sim \text{Gamma}(\alpha_1+\alpha_2, \lambda).$$ ← $\boxed{\text{intuition}}$



Proof. For $z\geq0$,

$$f_Z(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^z \overset{z^{\alpha_1-1}(\frac{x}{z})^{\alpha_1-1}}{x^{\alpha_1-1}}\overset{z^{\alpha_2-1}(1-\frac{x}{z})^{\alpha_2-1}}{(z-x)^{\alpha_2-1}}e^{-\overset{\lambda(x+(z-x))}{\boxed{\lambda z}}}dx$$

Let $y=x/z$ $\Rightarrow x=zy$ $\frac{dx}{dy}=z$ $\Rightarrow dx=zdy$

$$\stackrel{\downarrow}{=} \frac{\lambda^{\alpha_1+\alpha_2}e^{-\lambda z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 z^{(\alpha_1-1)+(\alpha_2\cancel{-1})\cancel{+1}}\underbrace{y^{\alpha_1-1}(1-y)^{\alpha_2-1}}_{\text{Beta function (LNp.6-27)}}dy$$

$$= \underbrace{\frac{\lambda^{\alpha_1+\alpha_2}z^{(\alpha_1+\alpha_2)-1}e^{-\lambda z}}{\cancel{\Gamma(\alpha_1)\Gamma(\alpha_2)}} \times \frac{\cancel{\Gamma(\alpha_1)\Gamma(\alpha_2)}}{\Gamma(\alpha_1+\alpha_2)}}_{\text{pdf of Gamma}(\alpha_1+\alpha_2,\lambda)}.$$

and $f_Z(z) = 0$, for $z < 0$.

- ▫ Corollary. If $X_1, \ldots, X_n$ are independent, and

$$X_i\sim \text{Gamma}(\alpha_i, \lambda),\ i=1, \ldots, n,$$

then $X_1 + \cdots + X_n \sim \text{Gamma}(\alpha_1+\cdots+\alpha_n, \lambda)$. ← $\boxed{\text{intuition}}$

Proof. By induction (exercise).

- ▫ Corollary. If $X_1, \ldots, X_n$ are independent, and

$$X_i \sim \text{Exponential}(\lambda),\ i=1, \ldots, n,$$ → Gamma$(1,\lambda)$

then $X_1 + \cdots + X_n \sim \text{Gamma}(n, \lambda)$. ← $\boxed{\text{intuition}}$

Proof. (exercise).

check Textbook Sec. 6.3.3

■ **Theorem.** If $X_1, \ldots, X_n$ are independent, and

$$X_i \sim \text{Normal}(\mu_i, \sigma_i^2),\ i=1, \ldots, n,$$

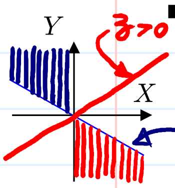
then $X_1 + \cdots + X_n \sim \text{Normal}(\mu_1+\cdots+\mu_n,\ \sigma_1^2+\cdots+\sigma_n^2)$.

$E(X_1+\cdots+X_n) = E(X_1)+\cdots+E(X_n)$

$Var(X_1+\cdots+X_n) \overset{\perp}{=} Var(X_1)+\cdots+Var(X_n)$

(Chapter 7)

Proof. (exercise).



$y/x = z$ $(z<0)$

$\Rightarrow y = xz$

■ **Example.** X and Y are random variables with joint pdf $f(x, y)$. Find the distribution of $Z=Y/X$. $= Y \times \frac{1}{X}$

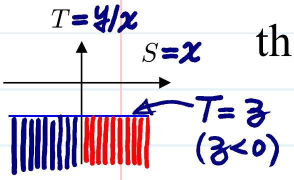
X & Y: continuous r.v.'s

□ Let $Q_z = \{(x,y) : y/x \le z\}$

$\textcircled{A} = \{(x,y) : x<0, y \ge zx\}$

$\cup \{(x,y) : x>0, y \le zx\}$

B: $\{Z \le z\}$



then, $\underset{P(Z\le z)}{F_Z(z)} = \int\int_{Q_z} f(x,y)\,dxdy$

$$= \int_{-\infty}^{0}\int_{xz}^{\infty} + \int_0^\infty \int_{-\infty}^{xz} f(x,y)\,dydx$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{z} + \int_0^\infty \int_{-\infty}^{z} f(s,st)\boxed{|s|}\,dtds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z} |s| f(s,st)\,dtds$$

$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} |s| f(s,st)\,dsdt$$

$|J|$

$\left(\text{set} \begin{cases} x = s \\ y = st \end{cases}\right)$

$s=x,\ t=y/x$

$J = \begin{vmatrix} \frac{dx}{ds} & \frac{dx}{dt} \\ \frac{dy}{ds} & \frac{dy}{dt} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ t & s \end{vmatrix} = s$

$dxdy = |s|\,dsdt$

and, $f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} |x| f(\underset{分母}{x}, \underset{分子}{zx})\,dx$

□ When X and Y are independent,

$$f(x, y) = f_X(x)f_Y(y).$$

So, $F_Z(z) = \int_{-\infty}^{z}\int_{-\infty}^{\infty} |s| f_X(s)f_Y(st)\,dsdt$ — (*)

and, $f_Z(z) = \int_{-\infty}^{\infty} |x| \underset{分母}{f_X(x)} \underset{分子}{f_Y(zx)}\,dx$

Hint.
$Z = XY$
$= Y/(X^{-1})$
Find the joint pdf of (X^{-1}, Y)

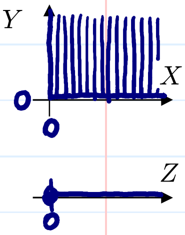
□ (exercise) $Z=XY$

□ If X and Y are independent,

$$X \sim \text{exponential}(\lambda_1), \text{ and } Y \sim \text{exponential}(\lambda_2),$$

e.g. ratio of 2 lifetimes

Let $Z=Y/X$. The pdf of Z is



$$f_Z(z) = \int_0^\infty x\left(\lambda_1 e^{-\lambda_1 x}\right)\left[\lambda_2 e^{-\lambda_2 (xz)}\right]\,dx$$

$$= \frac{\lambda_1\lambda_2\Gamma(2)}{(\lambda_1+\lambda_2 z)^2}\int_0^\infty \underbrace{\frac{(\lambda_1+\lambda_2 z)^2}{\Gamma(2)}x^{2-1}e^{-(\lambda_1+\lambda_2 z)x}}_{\text{pdf of Gamma}(2,\,(\lambda_1+\lambda_2)z)}\,dx$$

$$= \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2 z)^2}$$

for $z \ge 0$, and 0 for $z < 0$.

a special case of the method of cdf (see proof of the Thm)

And, the cdf of Z is

$$F_Z(z) = \int_0^z f_Z(t)\,dt = \int_0^z \frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2 t)^2}\,dt$$

$$= -\frac{\lambda_1\lambda_2}{\lambda_2}(\lambda_1+\lambda_2 t)^{-1}\Big|_0^z = 1-\frac{\lambda_1}{\lambda_1+\lambda_2 z} = \begin{cases} 1, & z\uparrow\infty \\ 0, & z\downarrow 0\end{cases}$$

for $z \geq 0$, and 0 for $z<0$.

➤ Method of probability density function

- **Theorem.** Let $\mathbf{X}=(X_1, \ldots, X_{\boxed{n}})$ be continuous random variables with the joint pdf $f_{\mathbf{X}}(x_1, \ldots, x_n)$. Let

$$\mathbf{Y}=(Y_1, \ldots, Y_{\boxed{n}}) = g(\mathbf{X}),$$

$(g_1, g_2, \cdots, g_n)$, $g_i: \mathbb{R}^n \to \mathbb{R}^1$

$Y_i = g_i(X_1, \cdots, X_n)$

where ① g is 1-to-1, so that its inverse exists and is denoted by

$$\mathbf{x}=g^{-1}(\mathbf{y}) = w(\mathbf{y}) = (w_1(\mathbf{y}), w_2(\mathbf{y}), \ldots, w_n(\mathbf{y})).$$

$w_i: \mathbb{R}^n \to \mathbb{R}^1$

$X_i = w_i(Y_1, \cdots, Y_n)$

Assume ② w have continuous partial derivatives. Let

Jacobin

determinant

$$J(y_1,\cdots,y_n) = J = \begin{vmatrix} \frac{\partial w_1(\mathbf{y})}{\partial y_1} & \frac{\partial w_1(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_1(\mathbf{y})}{\partial y_n} \\ \frac{\partial w_2(\mathbf{y})}{\partial y_1} & \frac{\partial w_2(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_2(\mathbf{y})}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial w_n(\mathbf{y})}{\partial y_1} & \frac{\partial w_n(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_n(\mathbf{y})}{\partial y_n} \end{vmatrix}_{n\times n} = \left|\left[\frac{\partial w_i}{\partial y_j}\right]\right|$$

Note. It becomes J^{-1} if use $\left|\left[\partial g_i/\partial x_j\right]\right|$

Then $f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(g^{-1}(\mathbf{y})) \times |J|$, ←c.f.→ Thm in LNp.6-10~11

absolute value of J

for $\mathbf{y}$ s.t. $\mathbf{y}=g(\mathbf{x})$ for some $\boxed{\mathbf{x}}$ and $f_{\mathbf{Y}}(\mathbf{y})=0$, otherwise.

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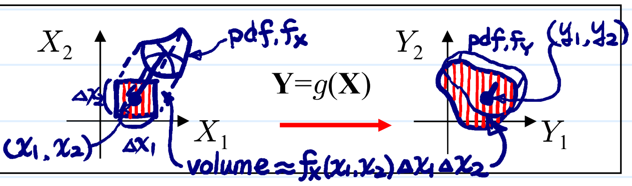
Recall. The question in LNp.6-11

(**Q**: What is the role of $|J|$?)

why absolute value?



Proof. $F_{\mathbf{Y}}(y_1,\ldots,y_n) = \int_{-\infty}^{y_1}\cdots\int_{-\infty}^{y_n} f_{\mathbf{Y}}(t_1,\ldots,t_n)\,dt_n\cdots dt_1$

$$= \int\cdots\int_{\substack{(x_1,\ldots,x_n):\\ g_1(x_1,\ldots,x_n)\leq y_1 \\ \cdots \\ g_n(x_1,\ldots,x_n)\leq y_n}} f_{\mathbf{X}}(x_1,\ldots,x_n)\,dx_n\cdots dx_1.$$

$Y_1 = g_1(x_1,\ldots,x_n) \leq y_1$, $Y_n = g_n(x_1,\ldots,x_n)\leq y_n$

method of cdf check 1~3 in LNp.7-31

$\mathbf{X} = g^{-1}(\mathbf{Y})$

$\int\cdots\int_S f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$

$= \int\cdots\int_{g(S)} f_{\mathbf{X}}(g^{-1}(\mathbf{y}))|J(\mathbf{y})|d\mathbf{y}$

$\|$

$\int_{-\infty}^{y_1}\cdots\int_{-\infty}^{y_n}$

It then follows from an exercise in advanced calculus that

$$f_{\mathbf{Y}}(y_1,\ldots,y_n) = \frac{\partial^n}{\partial y_1\cdots\partial y_n}F_{\mathbf{Y}}(y_1,\ldots,y_n)$$

$$= f_{\mathbf{X}}(\underbrace{w_1(\mathbf{y}),\ldots,w_n(\mathbf{y})}_{g^{-1}(\mathbf{y})}) \times |J|.$$

change of variables

Functions of $\mathbf{X}$

- **Remark.** When the dimensionality of $\mathbf{Y}$ (denoted by k) is less than n, we can choose another $n-k$ transformations $\mathbf{Z}$ such that

$$\underbrace{(\overset{k\text{-dim}}{\mathbf{Y}}, \overset{(n-k)\text{-dim}}{\mathbf{Z}})}_{n\text{-dim}} = g(\mathbf{X})$$

$\mathbb{R}^n \to \mathbb{R}^n$, g^{-1} exists, g^{-1}: continuously differentiable

satisfy the assumptions in above theorem.