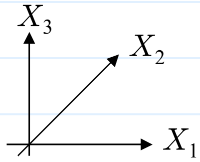
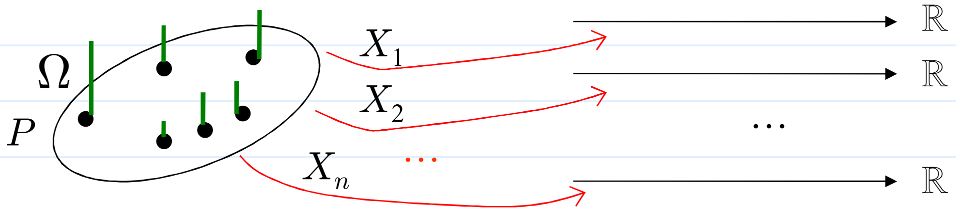
# Jointly Distributed Random Variables

- Recall. In Chapters 4 and 5, focus on *univariate* random variable.
  - However, often a single experiment will have more than one random variables which are of interest.




  - Definition. Given a sample space $\Omega$ and a probability measure $P$ defined on the subsets of $\Omega$, random variables

$$X_1, X_2, \ldots, X_n : \Omega \to \mathbb{R}$$

are said to be *jointly distributed*.

    - We can regard $n$ jointly distributed r.v.'s as a *random vector* $\mathbf{X}=(X_1, \ldots, X_n): \Omega \to \mathbb{R}^n$.

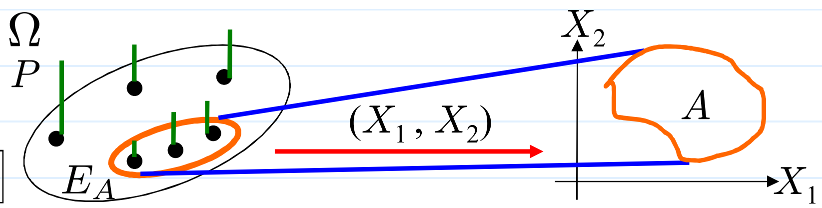




- **Q**: For $A\subset\mathbb{R}^n$, how to define the probability of $\{\mathbf{X}\in A\}$ from $P$?

$A$ occurs $\Leftrightarrow$ $E_A$ occurs

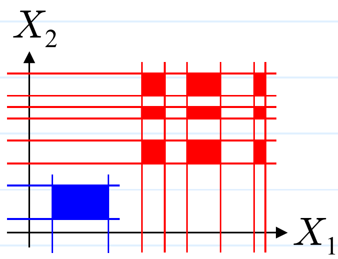
$P_{X_1,X_2}(A) = P(E_A)$




$P_{X_1,X_2}(A)$ $=??$

  - For $A\subset\mathbb{R}^n$,

$$P_{X_1,\ldots,X_n}(A) = P(\{\omega\in\Omega \mid (X_1(\omega),\ldots,X_n(\omega))\in A\})$$

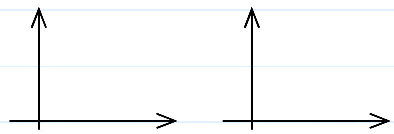



  - For $A_i\subset\mathbb{R}$, $i=1, \ldots, n$,

$$P_{X_1,\ldots,X_n}(X_1\in A_1,\cdots,X_n\in A_n) = P(\{\omega\in\Omega \mid X_1(\omega)\in A_1\}\cap\cdots\cap\{\omega\in\Omega \mid X_n(\omega)\in A_n\})$$

  - Definition. The probability measure of $\mathbf{X}$ ($P_{\mathbf{X}}$, defined on subsets of $\mathbb{R}^n$) is called the *joint distribution* of $X_1, \ldots, X_n$. The probability measure of $X_i$ ($P_{X_i}$, defined on subsets of $\mathbb{R}$) is called the *marginal distribution* of $X_i$.

- **Q**: Why need joint distribution? Why are marginal distributions not enough?



  - Example (Coin Tossing, Toss a fair coin 3 times, LNp.5-3).

| $X_2$: # of head on 1[st] toss | $X_1$: total # of heads | | | |
|---|---|---|---|---|
| | **0** (1/8) | **1** (3/8) | **2** (3/8) | **3** (1/8) |
| **0** (1/2) | 1/8 [1/16] | 2/8 [3/16] | 1/8 [3/16] | 0 [1/16] |
| **1** (1/2) | 0 [1/16] | 1/8 [3/16] | 2/8 [3/16] | 1/8 [1/16] |

    - blue numbers: joint distribution of $X_1$ and $X_2$
    - (black numbers): marginal distributions
    - [red numbers]: joint distribution of another $(X_1', X_2')$
    - Some findings:
      - When joint distribution is given, its corresponding marginal distributions are known, e.g.,
        - $P(X_1=i)=P(X_1=i, X_2=0)+P(X_1=i, X_2=1)$, $i=0, 1, 2, 3$.

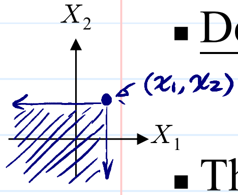




      - $(X_1, X_2)$ and $(X_1', X_2')$ have identical marginal distributions but different joint distributions.
        - When the marginal distributions are given, the corresponding joint distribution is still unknown. There could be many possible different joint distributions. (A special case: $X_1, \ldots, X_n$ are independent.)
      - Joint distribution offers more information, e.g.,
        - When not observing $X_1$, the distribution of $X_2$ is: $P(X_2=0)=1/2$, $P(X_2=1)=1/2$ $\Rightarrow$ marginal distribution
        - When $X_1$ was observed, say $X_1=1$, the distribution of $X_2$ is: $P(X_2=0|X_1=1)=(2/8)/(3/8)=2/3$ and $P(X_2=1|X_1=1)=(1/8)/(3/8)=1/3$ $\Rightarrow$ the calculation requires the knowing of joint distribution

- We can characterize the joint distribution of **X** in terms of its
  1. Joint Cumulative Distribution Function (joint cdf)
  2. Joint Probability Mass (Density) Function (joint pmf or pdf)
  3. Joint Moment Generating Function (joint mgf, Chapter 7)

## ➢ Joint Cumulative Distribution Function




- **Definition.** The joint cdf of $\mathbf{X}=(X_1, \ldots, X_n)$ is defined as

$$F_{\mathbf{X}}(x_1, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n).$$

- **Theorem.** Suppose that $F_{\mathbf{X}}$ is a joint cdf. Then,

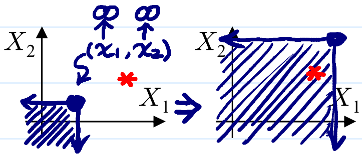
(i) $0 \le F_{\mathbf{X}}(x_1, \ldots, x_n) \le 1$, for $-\infty < x_i < \infty$, $i=1, \ldots, n$.

(ii) $\lim_{x_1, x_2, \cdots, x_n \to \infty} F_{\mathbf{X}}(x_1, \ldots, x_n) = 1$

Proof. Let $z_{im} \uparrow \infty$, $1 \le i \le n$.

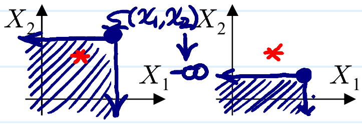
Let $A_m = (-\infty, z_{1m}] \times \cdots \times (-\infty, z_{nm}]$.

Then, $A_m \uparrow \mathbb{R}^n \Rightarrow \lim P(A_m) = P(\mathbb{R}^n) = 1$.




(iii) For any $i \in \{1, \ldots, n\}$,

$$\lim_{x_i \to -\infty} F_{\mathbf{X}}(x_1, \ldots, x_n) = 0.$$



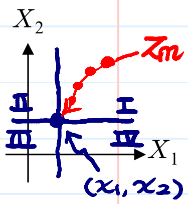

Proof. Let $z_{im} \downarrow -\infty$, for some $i$.

Let $A_m = (-\infty, x_1] \times \cdots \times (-\infty, z_{im}] \times \cdots \times (-\infty, x_n]$

Then, $A_m \downarrow \emptyset \Rightarrow \lim P(A_m) = P(\emptyset) = 0$.

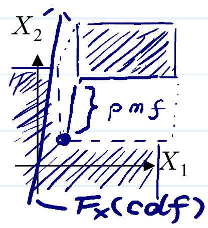


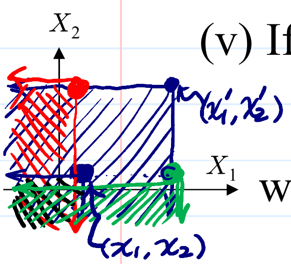





(iv) $F_{\mathbf{X}}$ is continuous from the right with respect to each of the coordinates, or any subset of them jointly, i.e., if $\mathbf{x}=(x_1, \ldots, x_n)$ and $\mathbf{z}_m=(z_{1m}, \ldots, z_{nm})$ such that $\mathbf{z}_m \downarrow \mathbf{x}$, then

$$F_{\mathbf{X}}(\mathbf{z}_m) \downarrow F_{\mathbf{X}}(\mathbf{x}).$$



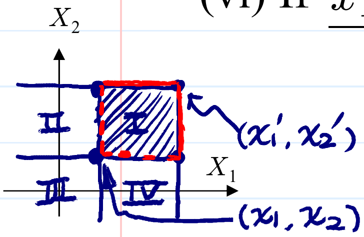

(v) If $x_i \le x_i'$, $i = 1, \ldots, n$, then

$$F_{\mathbf{X}}(x_1, \ldots, x_n) \le F_{\mathbf{X}}(t_1, \ldots, t_n) \le F_{\mathbf{X}}(x_1', \ldots, x_n').$$

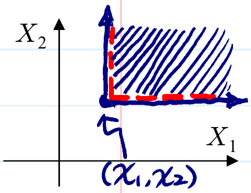
where $t_i \in \{x_i, x_i'\}$, $i = 1, 2, \ldots, n$. When $n=2$, we have

$$F_{X_1,X_2}(x_1, x_2) \le \left\{ \begin{array}{c} F_{X_1,X_2}(x_1, x_2') \\ F_{X_1,X_2}(x_1', x_2) \end{array} \right\} \le F_{X_1,X_2}(x_1', x_2').$$

(vi) If $x_1 \le x_1'$ and $x_2 \le x_2'$, then



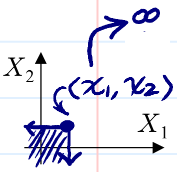

$$\begin{aligned} &P(x_1 < X_1 \le x_1', x_2 < X_2 \le x_2') \\ &= F_{X_1,X_2}(x_1', x_2') - F_{X_1,X_2}(x_1, x_2') \\ &\quad - F_{X_1,X_2}(x_1', x_2) + F_{X_1,X_2}(x_1, x_2). \end{aligned}$$

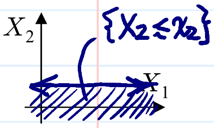



In particular, let $x_1' \uparrow \infty$ and $x_2' \uparrow \infty$, we get

$$\begin{aligned} & P(x_1 < X_1 < \infty, x_2 < X_2 < \infty) \\ & \quad = \quad 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1, X_2}(x_1, x_2). \end{aligned}$$

(vii) The joint cdf of $X_1, \ldots, X_k$, $k < n$, is




$$\begin{aligned} F_{X_1,\ldots,X_k}(x_1,\ldots,x_k) &= P(X_1 \le x_1, \ldots, X_k \le x_k) \\ &= P(X_1 \le x_1, \ldots, X_k \le x_k, \\ & \qquad -\infty < X_{k+1} < \infty, \ldots, -\infty < X_n < \infty) \\ &= \lim_{x_{k+1}, x_{k+2}, \cdots, x_n \to \infty} F_{\mathbf{X}}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n). \end{aligned}$$

In particular, the marginal cdf of $X_1$ is

$$\begin{aligned} F_{X_1}(x) &= P(X_1 \le x) \\ &= \lim_{x_2, x_3, \cdots, x_n \to \infty} F_{\mathbf{X}}(x, x_2, x_3, \ldots, x_n). \end{aligned}$$

- Theorem. A function $F_{\mathbf{X}}(x_1, \ldots, x_n)$ can be a joint cdf if $F_{\mathbf{X}}$ satisfies (i)-(v) in the previous theorem.





## ➢ Joint Probability Mass Function

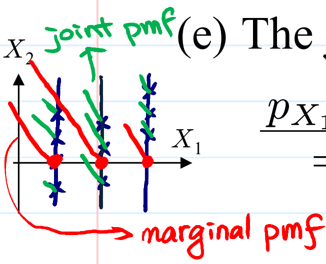
- Definition. Suppose that $X_1, \ldots, X_n$ are discrete random variables. The joint pmf of $\mathbf{X}=(X_1, \ldots, X_n)$ is defined as
$$p_{\mathbf{X}}(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n).$$
- Theorem. Suppose that $p_{\mathbf{X}}$ is a joint pmf. Then,

(a) $p_{\mathbf{X}}(x_1, \ldots, x_n) \ge 0$, for $-\infty < x_i < \infty$, $i = 1, \ldots, n$.

(b) There exists a finite or countably infinite set $\mathcal{X} \subset \mathbb{R}^n$ such that $p_{\mathbf{X}}(x_1, \ldots, x_n) = 0$, for $(x_1, \ldots, x_n) \notin \mathcal{X}$.

(c) $\sum_{\mathbf{x} \in \mathcal{X}} p_{\mathbf{X}}(\mathbf{x}) = 1$, where $\mathbf{x} = (x_1, \ldots, x_n)$.

(d) For $A \subset \mathbb{R}^n$, $P(\mathbf{X} \in A) = \sum_{\mathbf{x} \in A \cap \mathcal{X}} p_{\mathbf{X}}(\mathbf{x})$.



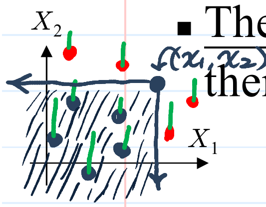

(e) The joint pmf of $X_1, \ldots, X_k$, $k < n$, is

$$\begin{aligned} p_{X_1,\ldots,X_k}(x_1,\ldots,x_k) &= P(X_1 = x_1, \ldots, X_k = x_k) \\ &= P(X_1 = x_1, \ldots, X_k = x_k, \\ & \qquad -\infty < X_{k+1} < \infty, \ldots, -\infty < X_n < \infty) \\ &= \sum_{\substack{(x_1,\ldots,x_n) \in \mathcal{X} \\ -\infty < x_{k+1} < \infty, \ldots, -\infty < x_n < \infty}} p_{\mathbf{X}}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n). \end{aligned}$$

In particular, the marginal pmf of $X_1$ is

$$p_{X_1}(x) = P(X_1 = x) = \sum_{\substack{(x, x_2, \ldots, x_n) \in \mathcal{X} \\ -\infty < x_2 < \infty, \ldots, -\infty < x_n < \infty}} p_{\mathbf{X}}(x, x_2, x_3, \ldots, x_n).$$

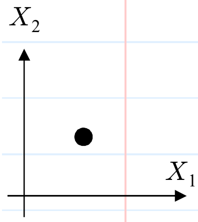
- Theorem. A function $p_{\mathbf{X}}(x_1, \ldots, x_n)$ can be a joint pmf if $p_{\mathbf{X}}$ satisfies (a)-(c) in the previous theorem.




- Theorem. If $F_{\mathbf{X}}$ and $p_{\mathbf{X}}$ are the joint cdf and joint pmf of $\mathbf{X}$, then

$$F_{\mathbf{X}}(x_1, \ldots, x_n) = \sum_{\substack{(t_1, \ldots, t_n) \in \mathcal{X} \\ t_1 \le x_1, \ldots, t_n \le x_n}} p_{\mathbf{X}}(t_1, \ldots, t_n).$$

To derive $p_{\mathbf{X}}$ from $F_{\mathbf{X}}$, take $n$=2 to illustrate:




$$p_{\mathbf{X}}(x_1, x_2) = \lim_{m \to \infty} P\left(x_1 - \frac{1}{m} < X_1 \le x_1 + \frac{1}{m}, x_2 - \frac{1}{m} < X_2 \le x_2 + \frac{1}{m}\right)$$

$$= \lim_{m \to \infty} \Big[F_{\mathbf{X}}(x_1 + 1/m, x_2 + 1/m) - F_{\mathbf{X}}(x_1 + 1/m, x_2 - 1/m) - F_{\mathbf{X}}(x_1 - 1/m, x_2 + 1/m) + F_{\mathbf{X}}(x_1 - 1/m, x_2 - 1/m)\Big]$$

$$= F_{\mathbf{X}}(x_1, x_2) - F_{\mathbf{X}}(x_1, x_2-) - F_{\mathbf{X}}(x_1-, x_2) + F_{\mathbf{X}}(x_1-, x_2-)$$



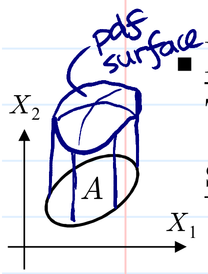


## ➢ Joint Probability Density Function

- Definition. A function $f_{\mathbf{X}}(x_1, \ldots, x_n)$ can be a joint pdf if

(1) $f_{\mathbf{X}}(x_1, \ldots, x_n) \ge 0$, for $-\infty < x_i < \infty$, $i$=1, …, $n$.

(2) $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = 1.$

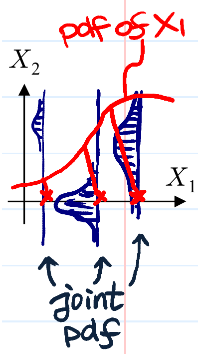



- Definition. Suppose that $X_1, \ldots, X_n$ are continuous r.v.'s. The joint pdf of $\mathbf{X}$=($X_1, \ldots, X_n$) is a function $f_{\mathbf{X}}(x_1, \ldots, x_n)$ satisfying (1) and (2) above, and for any event $A \subset \mathbb{R}^n$,

$$P(\mathbf{X} \in A) = \int \cdots \int_A f_{\mathbf{X}}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

- Theorem. Suppose that $f_{\mathbf{X}}$ is the joint pdf of $\mathbf{X}$=($X_1, \ldots, X_n$). Then, the joint pdf of $X_1, \ldots, X_k$, $k < n$, is

$$f_{X_1, \ldots, X_k}(x_1, \ldots, x_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)\, dx_{k+1} \cdots dx_n.$$




In particular, the marginal pdf of $X_1$ is

$$f_{X_1}(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\mathbf{X}}(x, x_2, \ldots, x_n)\, dx_2 \cdots dx_n.$$

- Theorem. If $F_{\mathbf{X}}$ and $f_{\mathbf{X}}$ are the joint cdf and joint pdf of $\mathbf{X}$, then

$$F_{\mathbf{X}}(x_1, \ldots, x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f_{\mathbf{X}}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n, \quad \text{and}$$

$$f_{\mathbf{X}}(x_1, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{\mathbf{X}}(x_1, \ldots, x_n).$$

at the continuity points of $f_{\mathbf{X}}$.

- Examples.

➢ Experiment. Two balls are drawn without replacement from a box with one ball labeled 1,

two balls labeled 2,

three balls labeled 3.

Let $X$ = label on the 1st ball drawn,

$Y$ = label on the 2nd ball drawn.

- The joint pmf and marginal pmfs of $(X, Y)$ are





| $p(x, y)$ | | $X$ = 1 | 2 | 3 | $p_Y(y)$ |
|---|---|---|---|---|---|
| $Y$ | 1 | 0 | 2/30 | 3/30 | 1/6 |
| | 2 | 2/30 | 2/30 | 6/30 | 2/6 |
| | 3 | 3/30 | 6/30 | 6/30 | 3/6 |
| $p_X(x)$ | | 1/6 | 2/6 | 3/6 | |

**Q**: The balls are drawn without replacement. Why do $X$ (from 1st ball) and $Y$ (from 2nd ball) have same marginal distributions?

- **Q**: $P(|X-Y|=1)$=??

$$P(|X-Y|=1) = P(X=1, Y=2) + P(X=2, Y=1) + P(X=2, Y=3) + P(X=3, Y=2) = 8/15.$$

- **Q**: What are the joint pmf and marginal pmfs of $(X, Y)$ if the balls are drawn with replacement (LNp. 4-6)?

| $p(x, y)$ | | $X$ = 1 | 2 | 3 | $p_Y(y)$ |
|---|---|---|---|---|---|
| $Y$ | 1 | 1/36 | 2/36 | 3/36 | 1/6 |
| | 2 | 2/36 | 4/36 | 6/36 | 2/6 |
| | 3 | 3/36 | 6/36 | 9/36 | 3/6 |
| $p_X(x)$ | | 1/6 | 2/6 | 3/6 | |

## ➢ Multinomial Distribution

- Recall. Partitions
  - If $n \geq 1$ and $n_1, \ldots, n_m \geq 0$ are integers for which
  $$n_1 + \cdots + n_m = n,$$
  then a set of $n$ elements may be partitioned into $m$ subsets of sizes $n_1, \ldots, n_m$ in
  $$\binom{n}{n_1, \cdots, n_m} = \frac{n!}{n_1! \times \cdots \times n_m!} \quad \text{ways.}$$
  - Example (LNp.2-8) : MISSISSIPPI
  $$\binom{11}{4,1,2,4} = \frac{11!}{4!1!2!4!}.$$
- Example (Die Rolling).
  - **Q**: If a balanced (6-sided) die is rolled 12 times, $P$(each face appears twice)=??
  - Sample space of rolling the die once (basic experiment):
  $$\Omega_0 = \{1, 2, 3, 4, 5, 6\}.$$





  - The sample space for the 12 trials is
  $$\Omega = \Omega_0 \times \cdots \times \Omega_0 = \Omega_0^{12}$$
  An outcome $\omega \in \Omega$ is $\omega = (i_1, i_2, \ldots, i_{12})$, where $1 \leq i_1, \ldots, i_{12} \leq 6$.
  - There are $6^{12}$ possible outcomes in $\Omega$, i.e., $\#\Omega = 6^{12}$.
  - Among all possible outcomes, there are $\binom{12}{2,2,2,2,2,2} = \frac{12!}{(2!)^6}$ of which each face appears twice.
  - $P(\text{each face appears twice}) = \frac{12!}{(2!)^6} \left(\frac{1}{6}\right)^{12}.$
- Generalization.
  - Consider a basic experiment which can result in one of $m$ types of outcomes. Denote its sample space as
  $$\Omega_0 = \{1, 2, \ldots, m\}.$$
  Let $p_i = P$(outcome $i$ appears in a basic experiment), then, (i) $p_1, \ldots, p_m \geq 0$, and (ii) $p_1 + \cdots + p_m = 1$.

- Repeat the basic experiment $n$ times. Then, the sample space for the $n$ trials is

$$\Omega = \Omega_0 \times \cdots \times \Omega_0 = \Omega_0^n$$

Let $X_i$ = # of trials with outcome $i$, $i$=1, ..., $m$,

Then, (i) $X_1, \ldots, X_m: \Omega \to \mathbb{R}$, and

(ii) $X_1 + \cdots + X_m = n$.

- The joint pmf of $X_1, \ldots, X_m$ is

$$\begin{aligned} p_{\mathbf{X}}(x_1, \ldots, x_m) &= P(X_1 = x_1, \ldots, X_m = x_m) \\ &= \binom{n}{x_1, \cdots, x_m} p_1^{x_1} \times \cdots \times p_m^{x_m}. \end{aligned}$$

for $x_1, \ldots, x_m \geq 0$ and $x_1 + \cdots + x_m = n$.

Proof. The probability of any sequence with $x_i$ $i$'s is

$$p_1^{x_1} \times \cdots \times p_m^{x_m},$$

and there are

$$\binom{n}{x_1, \cdots, x_m}$$

such sequences.





- The distribution of a random vector $\mathbf{X}=(X_1, \ldots, X_m)$ with the above joint pmf is called the *multinomial* distribution with parameters $n$, $m$, and $p_1, \ldots, p_m$, denoted by Multinomial($n, m, p_1, \ldots, p_m$).
  - The multinomial distribution is called after the Multinomial Theorem:

$$(a_1 + \cdots + a_m)^n = \sum_{\substack{x_i \in \{0, \ldots, n\};\ i=1, \ldots, m \\ x_1 + \cdots + x_m = n}} \binom{n}{x_1, \cdots, x_m} a_1^{x_1} \times \cdots \times a_m^{x_m}.$$

  - It is a generalization of the binomial distribution from 2 types of outcomes to $m$ types of outcomes.
- Some Properties.
  - Because $X_i = n - (X_1 + \cdots + X_{i-1} + X_{i+1} + \cdots + X_m)$, and

$$p_i = 1 - (p_1 + \cdots + p_{i-1} + p_{i+1} + \cdots + p_m),$$

WLOG, we can write

$$(X_1, \ldots, X_{m-1}, X_m) \to (X_1, \ldots, X_{m-1}, n-(X_1 + \cdots + X_{m-1}))$$

- **Marginal Distribution.** Suppose that $(X_1, \ldots, X_m) \sim \text{Multinomial}(n, m, p_1, \ldots, p_k, p_{k+1}, \ldots, p_m)$. For $1 \le k < m$, the distribution of

$$(X_1, \ldots, X_k, X_{k+1} + \cdots + X_m)$$

is $\text{Multinomial}(n, k+1, p_1, \ldots, p_k, p_{k+1} + \cdots + p_m)$. In particular, $X_i \sim \text{Binomial}(n, p_i)$

- **Mean and Variance.**

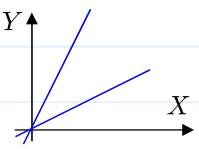
$$E(X_i) = np_i \text{ and } Var(X_i) = np_i(1-p_i)$$

for $i = 1, \ldots, m$.

➢ Example.

- Suppose that the joint pdf of 2 continuous r.v.'s $(X, Y)$ is

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)}, & x \ge 0,\ y \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$




**Q**: $P(Y \ge 2X \text{ or } X \ge 2Y) = ??$

- The event $\{Y \ge 2X\} \cup \{X \ge 2Y\}$ is



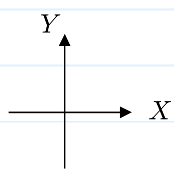


- So, $P(Y \ge 2X \text{ or } X \ge 2Y) = P(Y \ge 2X) + P(X \ge 2Y) = 2/3$ because

$$\begin{aligned} P(Y \ge 2X) &= \int_0^\infty \left[ \int_{2x}^\infty \lambda^2 e^{-\lambda(x+y)}\, dy \right] dx \\ &= \int_0^\infty -\lambda e^{-\lambda(x+y)} \Big|_{y=2x}^{\infty} dx = \int_0^\infty \lambda e^{-3\lambda x}\, dx \\ &= (-1/3) e^{-3\lambda x} \Big|_{x=0}^{\infty} = 1/3. \end{aligned}$$

and similarly, we can get $P(X \ge 2Y) = 1/3$ (exercise).

➢ Example. Consider two continuous r.v.'s $X$ and $Y$.




- Uniform Distribution over a region $D$. If $D \subset \mathbb{R}^2$ and $0 < \alpha = \text{Area}(D) < \infty$, then

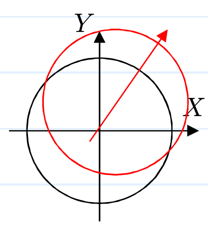
$$f(x,y) = c \cdot \mathbf{1}_D(x,y)$$

is a joint pdf when $c = 1/\alpha$, called the uniform pdf over $D$.

- Let $D = \{(x, y): x^2 + y^2 \le 1\}$, then $\alpha = \text{Area}(D) = \pi$ and

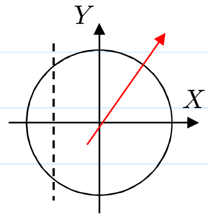
$$f(x,y) = \frac{1}{\pi} \mathbf{1}_D(x,y)$$

is a joint pdf.



- Marginal distribution. The marginal pdf of $X$ is

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2}{\pi}\sqrt{1-x^2}$$



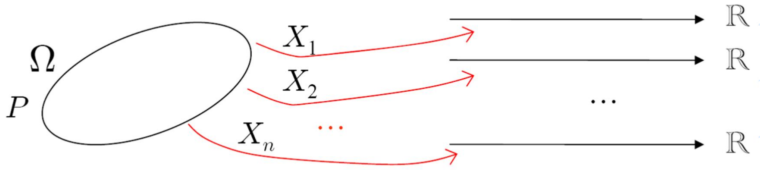

for $-1 \leq x \leq 1$, and $f_X(x)=0$, otherwise.
(exercise: Find the marginal distribution of $Y$.)

❖ **Reading**: textbook, Sec 6.1

## Independent Random Variables

- Recall.
  - ➢ If joint distribution is given, marginal distributions are known.
  - ➢ The converse statement does not hold in general.
  - ➢ However, when random variables are independent,

    marginal distributions + independence ⇒ joint distribution.








- Definition. The $n$ jointly distributed r.v.'s $X_1, \ldots, X_n$ are called (*mutually*) *independent* if and only if for *any* (measurable) sets $A_i \subset \mathbb{R}$, $i=1, \ldots, n$, the events

$$\{X_1 \in A_1\}, \ldots, \{X_n \in A_n\}$$

are (mutually) independent. That is,

$$P(X_{i_1} \in A_{i_1}, X_{i_2} \in A_{i_2}, \cdots, X_{i_k} \in A_{i_k}) = P(X_{i_1} \in A_{i_1}) \times P(X_{i_2} \in A_{i_2}) \times \cdots \times P(X_{i_k} \in A_{i_k}),$$

for *any* $1 \leq i_1 < i_2 < \cdots < i_k \leq n$; $k=2, \ldots, n$.

  - ➢ If $X_1, \ldots, X_n$ are independent, for $1 \leq k < n$,

$$P(X_{k+1} \in A_{k+1}, \ldots, X_n \in A_n | X_1 \in A_1, \ldots, X_k \in A_k) = P(X_{k+1} \in A_{k+1}, \ldots, X_n \in A_n)$$

provided that $P(X_1 \in A_1, \ldots, X_k \in A_k) > 0$.

  - In other words, the values of $X_1, \ldots, X_k$ do *not* carry any information about the distribution of $X_{k+1}, \ldots, X_n$.

- Theorem (Factorization Theorem). The random variables $\mathbf{X}=(X_1, \ldots, X_n)$ are independent if and only if one of the following conditions holds.

  (1) $F_{\mathbf{X}}(x_1, \ldots, x_n) = F_{X_1}(x_1) \times \cdots \times F_{X_n}(x_n)$, where $F_{\mathbf{X}}$ is the joint cdf of $\mathbf{X}$ and $F_{X_i}$ is the marginal cdf of $X_i$ for $i=1,\ldots,n$.

  (2) Suppose that $X_1, \ldots, X_n$ are discrete random variables. $p_{\mathbf{X}}(x_1, \ldots, x_n) = p_{X_1}(x_1) \times \cdots \times p_{X_n}(x_n)$, where $p_{\mathbf{X}}$ is the joint pmf of $\mathbf{X}$ and $p_{X_i}$ is the marginal pmf of $X_i$ for $i=1,\ldots,n$.

  (3) Suppose that $X_1, \ldots, X_n$ are continuous random variables. $f_{\mathbf{X}}(x_1, \ldots, x_n) = f_{X_1}(x_1) \times \cdots \times f_{X_n}(x_n)$, where $f_{\mathbf{X}}$ is the joint pdf of $\mathbf{X}$ and $f_{X_i}$ is the marginal pdf of $X_i$ for $i=1,\ldots,n$.

Proof.

independent $\Rightarrow$ (1).
$$\begin{aligned}F_{\mathbf{X}}(x_1, \ldots, x_n) &= P(X_1 \le x_1, \ldots, X_n \le x_n)\\ &= P(X_1 \in (-\infty, x_1], \ldots, X_n \in (-\infty, x_n])\\ &= P(X_1 \in (-\infty, x_1]) \times \cdots \times P(X_n \in (-\infty, x_n])\\ &= F_{X_1}(x_1) \times \cdots \times F_{X_n}(x_n)\end{aligned}$$

independent $\Leftarrow$ (1). Out of the scope of this couse so skip.





independent $\Rightarrow$ (2).
$$\begin{aligned}p_{\mathbf{X}}(x_1, \ldots, x_n) &= P(X_1 = x_1, \ldots, X_n = x_n)\\ &= P(X_1 \in \{x_1\}, \ldots, X_n \in \{x_n\})\\ &= P(X_1 \in \{x_1\}) \times \cdots \times P(X_n \in \{x_n\})\\ &= p_{X_1}(x_1) \times \cdots \times p_{X_n}(x_n)\end{aligned}$$

(2) $\Rightarrow$ (1).
$$\begin{aligned}F_{\mathbf{X}}(x_1, \ldots, x_n) &= \sum_{\substack{(t_1,\ldots,t_n)\in\mathcal{X}\\ t_1\le x_1,\ldots,t_n\le x_n}} p_{\mathbf{X}}(t_1, \ldots, t_n)\\ &= \sum_{\substack{(t_1,\ldots,t_n)\in\mathcal{X}\\ t_1\le x_1}} \cdots \sum_{\substack{(t_1,\ldots,t_n)\in\mathcal{X}\\ t_n\le x_n}} p_{X_1}(t_1) \times \cdots \times p_{X_n}(t_n)\\ &= \sum_{\substack{(t_1,\ldots,t_n)\in\mathcal{X}\\ t_1\le x_1}} p_{X_1}(t_1) \times \cdots \times \sum_{\substack{(t_1,\ldots,t_n)\in\mathcal{X}\\ t_n\le x_n}} p_{X_n}(t_n) = F_{X_1}(x_1) \times \cdots \times F_{X_n}(x_n)\end{aligned}$$

---

(3) $\Rightarrow$ (1).
$$\begin{aligned}F_{\mathbf{X}}(x_1, \ldots, x_n) &= \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f_{\mathbf{X}}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n\\ &= \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f_{X_1}(t_1) \times \cdots \times f_{X_n}(t_n)\, dt_1 \cdots dt_n\\ &= \int_{-\infty}^{x_1} f_{X_1}(t_1)\, dt_1 \times \cdots \times \int_{-\infty}^{x_n} f_{X_n}(t_n)\, dt_n = F_{X_1}(x_1) \times \cdots \times F_{X_n}(x_n)\end{aligned}$$

$\underline{(3) \Leftarrow (1).}$

$$
\begin{aligned}
\underline{f_{\mathbf{X}}(x_1,\ldots,x_n)} &= \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{\mathbf{X}}(x_1,\ldots,x_n).\\
&= \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{X_1}(x_1) \times \cdots \times F_{X_n}(x_n)\\
&= \frac{\partial}{\partial x_1} F_{X_1}(x_1) \times \cdots \times \frac{\partial}{\partial x_n} F_{X_n}(x_n) = \underline{f_{X_1}(x_1) \times \cdots \times f_{X_n}(x_n)}
\end{aligned}
$$

➢Remark. It follows from the Multiplication Law (LNp.4-11) that

$$
\begin{aligned}
F_{\mathbf{X}}(x_1,\ldots,x_n) &= P(X_1 \le x_1,\ldots,X_n \le x_n)\\
= \quad & P(X_1 \le x_1) && (= F_{X_1}(x_1))\\
& \times P(X_2 \le x_2 | X_1 \le x_1) && \left(\stackrel{?}{=} P(X_2 \le x_2) = F_{X_2}(x_2)\right)\\
& \times P(X_3 \le x_3 | X_1 \le x_1, X_2 \le x_2) && \left(\stackrel{?}{=} P(X_3 \le x_3) = F_{X_3}(x_3)\right)\\
& \times \cdots\\
& \times P(X_n \le x_n | X_1 \le x_1,\ldots,X_{n-1} \le x_{n-1}) && \left(\stackrel{?}{=} P(X_n \le x_n) = F_{X_n}(x_n)\right)
\end{aligned}
$$

The independence can be established *sequentially*.

➢Example. If $A_1, \ldots, A_n \subset \mathbf{\Omega}$ are independent events, then $\mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_n}$, are independent random variables. For example,





$$
\begin{aligned}
& P(\mathbf{1}_{A_1} = 1, \mathbf{1}_{A_2} = 0, \mathbf{1}_{A_3} = 1)\\
&= \quad P(A_1 \cap A_2^c \cap A_3) = P(A_1)P(A_2^c)P(A_3)\\
&= \quad P(\mathbf{1}_{A_1} = 1)P(\mathbf{1}_{A_2} = 0)P(\mathbf{1}_{A_3} = 1).
\end{aligned}
$$

➢Theorem. If $\mathbf{X}=(X_1, \ldots, X_n)$ are independent and

$Y_i = g_i(X_i)$, $i=1, \ldots, n$,

then

$Y_1, \ldots, Y_n$ are independent.

| generalization |
|---|
| $1 = i_0 < i_1 < \cdots < i_k = n$ |
| $Y_1 = g_1(X_1,\ldots,X_{i_1})$, |
| $Y_2 = g_2(X_{i_1+1},\ldots,X_{i_2})$, |
| $\cdots$ |
| $Y_k = g_k(X_{i_{k-1}+1},\ldots,X_{i_k})$. |

Proof.

Let $A_i(y) = \{x : g_i(x) \le y\}$, $i=1, \ldots, n$, then

$$
\begin{aligned}
F_{\mathbf{Y}}(y_1,\ldots,y_n) &= P(Y_1 \le y_1,\ldots,Y_n \le y_n)\\
&= P(X_1 \in A_1(y_1),\ldots,X_n \in A_n(y_n))\\
&= P(X_1 \in A_1(y_1)) \times \cdots \times P(X_n \in A_n(y_n))\\
&= P(Y_1 \le y_1) \times \cdots \times P(Y_n \le y_n)\\
&= F_{Y_1}(y_1) \times \cdots \times F_{Y_n}(y_n).
\end{aligned}
$$

- Theorem. $\mathbf{X}=(X_1, \ldots, X_n)$ are independent if and only if there exist univariate functions $g_i(x)$, $i=1, \ldots, n$, such that

  (a) when $X_1, \ldots, X_n$ are discrete r.v.'s with joint pmf $p_{\mathbf{X}}$,

  $$p_{\mathbf{X}}(x_1, \ldots, x_n) \propto g_1(x_1)\times\cdots\times g_n(x_n),\ -\infty<x_i<\infty,\ i=1,\ldots,n.$$

  (b) when $X_1, \ldots, X_n$ are continuous r.v.'s with joint pdf $f_{\mathbf{X}}$,

  $$f_{\mathbf{X}}(x_1, \ldots, x_n) \propto g_1(x_1)\times\cdots\times g_n(x_n),\ -\infty<x_i<\infty,\ i=1,\ldots,n.$$

Sketch of proof for (b).

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, x_2, \ldots, x_n)\, dx_2\cdots dx_n$$

$$\propto \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g_1(x_1)g_2(x_2)\cdots g_n(x_n)\, dx_2\cdots dx_n \propto g_1(x_1).$$

Similarly, $f_{X_2}(x_2) \propto g_2(x_2), \ldots, f_{X_n}(x_n) \propto g_n(x_n)$

$$\Rightarrow\ f_{X_1}(x_1)\cdots f_{X_n}(x_n) \propto g_1(x_1)\cdots g_n(x_n)$$

$$\Rightarrow\ f_{\mathbf{X}}(x_1, \ldots, x_n) \propto f_{X_1}(x_1)\cdots f_{X_n}(x_n)$$

$$\Rightarrow\ f_{\mathbf{X}}(x_1, \ldots, x_n) = c \cdot f_{X_1}(x_1)\cdots f_{X_n}(x_n)$$

for some constant $c$.





Because $\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, x_2, \ldots, x_n)\, dx_1\cdots dx_n = 1$, and

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{X_1}(x_1)\cdots f_{X_n}(x_n)\, dx_1\cdots dx_n = 1, \quad \Rightarrow c = 1.$$

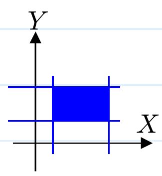
➢ Example.

- If the joint pdf of $(X, Y)$ is

  $$f(x,y) \propto e^{-2x}e^{-3y}, \quad 0<x<\infty, 0<y<\infty,$$

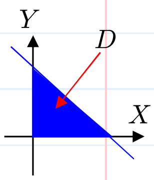
  and $f(x, y)=0$, otherwise, i.e.,

  $$f(x,y) \propto e^{-2x}e^{-3y}\mathbf{1}_{(0,\infty)}(x)\mathbf{1}_{(0,\infty)}(y),$$

  then $X$ and $Y$ are independent. Note that the region in which the joint pdf is nonzero can be expressed in the form $\{(x, y): x\in A, y\in B\}$.

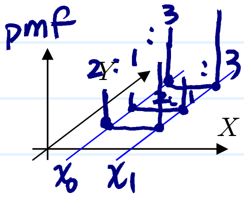
- Suppose that the joint pdf of $(X, Y)$ is

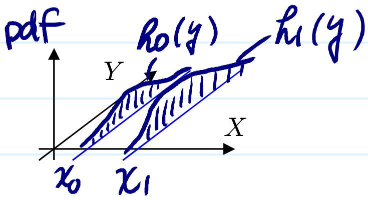
  
  

  $$f(x,y) \propto xy, \quad 0<x<1, 0<y<1, 0<x+y<1,$$

  and $f(x, y)=0$, otherwise, i.e., $f(x,y) \propto xy \cdot \mathbf{1}_D(x,y)$, $X$ and $Y$ are not independent.

➢ **Q**: For independent $X$ and $Y$, how should their joint pdf/pmf look like?




$\frac{h_1(y)}{h_0(y)}$ = a constant

❖ **Reading**: textbook, Sec 6.2

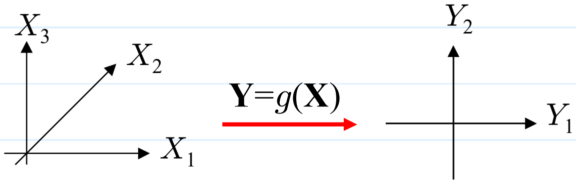
# Transformation

- **Q**: Given the joint distribution of $\mathbf{X}=(X_1, \ldots, X_n)$, how to find the distribution of $\mathbf{Y}=(Y_1, \ldots, Y_k)$, where

$$Y_1=g_1(X_1, \ldots, X_n) : \mathbb{R}^n \to \mathbb{R},$$
$$\ldots,$$
$$Y_k=g_k(X_1, \ldots, X_n) : \mathbb{R}^n \to \mathbb{R},$$

denoted by

$$\mathbf{Y}=g(\mathbf{X}),\ g: \mathbb{R}^n \to \mathbb{R}^k.$$








➢ The following methods are useful:

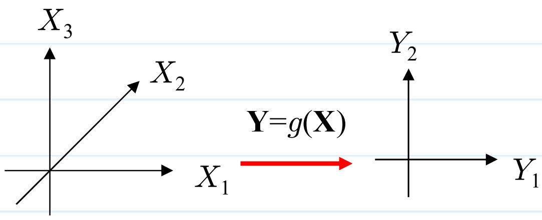
1. Method of Events ($\to$ pmf)
2. Method of Cumulative Distribution Function
3. Method of Probability Density Function
4. Method of Moment Generating Function (chapter 7)

➢ Method of Events

- Theorem. The distribution of $\mathbf{Y}$ is determined by the distribution of $\mathbf{X}$ as follows: for any event $B \subset \mathbb{R}^k$,

$$P_{\mathbf{Y}}(\mathbf{Y} \in B) = P_{\mathbf{X}}(\mathbf{X} \in A),$$

where $A = g^{-1}(B) \subset \mathbb{R}^n$.




- Example. Let $\mathbf{X}$ be a discrete random vector taking values

$$\mathbf{x}_i=(x_{1i}, x_{2i}, \ldots, x_{ni}),\ i=1, 2, \ldots,$$

(i.e., $\mathcal{X}=\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$) with joint pmf $p_{\mathbf{X}}$.

Then, $\mathbf{Y}=g(\mathbf{X})$ is also a discrete random vector.

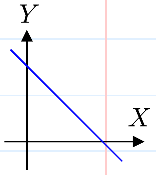
Suppose that $\mathbf{Y}$ takes values on $\mathbf{y}_j$, $j$= 1, 2, …. To determine the joint pmf of $\mathbf{Y}$, by taking $B$={$\mathbf{y}_j$}, we have

$$A = \{\mathbf{x}_i \in \mathcal{X} : g(\mathbf{x}_i) = \mathbf{y}_j\}$$

and hence, the joint pmf of $\mathbf{Y}$ is

$$p_{\mathbf{Y}}(\mathbf{y}_j) = P_{\mathbf{Y}}(\{\mathbf{y}_j\}) = P_{\mathbf{X}}(A) = \textstyle\sum_{\mathbf{x}_i \in A} p_{\mathbf{X}}(\mathbf{x}_i).$$

- Example. Let $X$ and $Y$ be random variables with the joint pmf $p(x, y)$. Find the distribution of $Z$=$X$+$Y$.




  - $\{Z=z\}$ = $\{(X, Y) \in \{(x, y): x+y=z\}\}$

$$p_Z(z) = P_Z(\{z\}) = P(X+Y=z) = \sum_{x \in \mathcal{X}_X} p(x, z-x).$$

  - When $X$ and $Y$ are independent,

$$p(x, y) = p_X(x)p_Y(y),$$

So,
$$p_Z(z) = \sum_{x \in \mathcal{X}_X} p_X(x)p_Y(z-x).$$

which is referred to as the *convolution* of $p_X$ and $p_Y$.

  - (Exercise) $Z$=$X$−$Y$

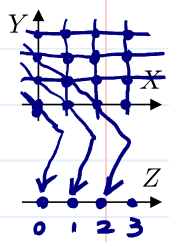




- Theorem. If $X$ and $Y$ are independent, and

$$X \sim \text{Poisson}(\lambda_1), \quad Y \sim \text{Poisson}(\lambda_2),$$

then $Z = X + Y \sim \text{Poisson}(\lambda_1+\lambda_2)$.




Proof. For $z$=0, 1, 2, …, the pmf $p_Z(z)$ of $Z$ is

$$p_Z(z) = \sum_{x=0}^{z} p_X(x)p_Y(z-x) = \sum_{x=0}^{z} \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^{z-x}}{(z-x)!}$$

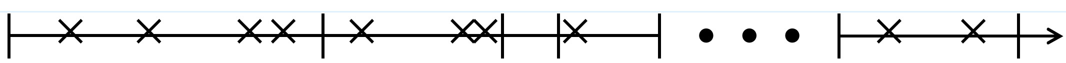
$$= \frac{e^{-(\lambda_1+\lambda_2)}}{z!}\left(\sum_{x=0}^{z}\frac{z!}{x!(z-x)!}\lambda_1^x\lambda_2^{z-x}\right) = \frac{e^{-(\lambda_1+\lambda_2)}}{z!}(\lambda_1+\lambda_2)^z.$$

  - Corollary. If $X_1, \ldots, X_n$ are independent, and

$$X_i \sim \text{Poisson}(\lambda_i),\ i=1, \ldots, n,$$

then $X_1+\cdots+X_n \sim \text{Poisson}(\lambda_1+\cdots+\lambda_n)$.

Proof. By induction (exercise).



## ➢ Method of cumulative distribution function

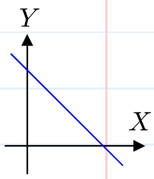
1. In the $(X_1, \ldots, X_n)$ space, find the region that corresponds to

$$\{Y_1 \leq y_1, \ldots, Y_k \leq y_k\}.$$

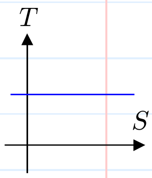
2. Find $F_{\mathbf{Y}}(y_1, \ldots, y_k) = P(Y_1 \leq y_1, \ldots, Y_k \leq y_k)$ by summing the joint pmf or integrating the joint pdf of $X_1, \ldots, X_n$ over the region identified in 1.

3. (for continuous case) Find the joint pdf of $\mathbf{Y}$ by differentiating $F_{\mathbf{Y}}(y_1, \ldots, y_k)$, i.e.,

$$f_{\mathbf{Y}}(y_1, \ldots, y_k) = \frac{\partial^k}{\partial y_1 \cdots \partial y_k} F_{\mathbf{Y}}(y_1, \ldots, y_k).$$




- **Example.** $X$ and $Y$ are random variables with joint pdf $f(x, y)$. Find the distribution of $Z = X + Y$.
  - $\{Z \leq z\} = \{(X, Y) \in \{(x, y): x+y \leq z\}\}$. So,




$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P(X + Y \leq z) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f(x, y)\, dy dx \\ &= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f(s, t-s)\, ds dt \qquad \left( \text{set} \begin{cases} x = s \\ y = t - s \end{cases} \right) \end{aligned}$$





and $f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f(x, z-x) dx$

- When $X$ and $Y$ are independent,

$$f(x, y) = f_X(x) f_Y(y).$$

So, 
$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y)\, dy dx \\ &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{z-x} f_Y(y)\, dy \right] f_X(x)\, dx \\ &= \int_{-\infty}^{\infty} F_Y(z-x) f_X(x)\, dx \end{aligned}$$

which is referred to as the *convolution* of $F_X$ and $F_Y$, and

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

which is referred to as the *convolution* of $f_X$ and $f_Y$.

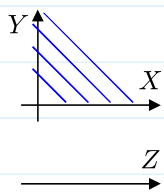
- (exercise) $Z = X - Y$.

- **Theorem.** If $X$ and $Y$ are independent, and

$X \sim \text{Gamma}(\alpha_1, \lambda)$, $Y \sim \text{Gamma}(\alpha_2, \lambda)$,

then

$$Z = X + Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \lambda).$$



Proof. For $z \geq 0$,

$$f_Z(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^z x^{\alpha_1-1}(z-x)^{\alpha_2-1}e^{-\lambda z}\, dx$$
$$= \frac{\lambda^{\alpha_1+\alpha_2}e^{-\lambda z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 z^{(\alpha_1-1)+(\alpha_2-1)+1} y^{\alpha_1-1}(1-y)^{\alpha_2-1}\, dy$$
$$= \frac{\lambda^{\alpha_1+\alpha_2} z^{(\alpha_1+\alpha_2)-1} e^{-\lambda z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \times \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}.$$

and $f_Z(z) = 0$, for $z < 0$.

- Corollary. If $X_1, \ldots, X_n$ are independent, and $X_i \sim$ Gamma$(\alpha_i, \lambda)$, $i=1, \ldots, n$, then $X_1 + \cdots + X_n \sim$ Gamma$(\alpha_1 + \cdots + \alpha_n, \lambda)$.

  Proof. By induction (exercise).

- Corollary. If $X_1, \ldots, X_n$ are independent, and $X_i \sim$ Exponential$(\lambda)$, $i=1, \ldots, n$, then $X_1 + \cdots + X_n \sim$ Gamma$(n, \lambda)$.

  Proof. (exercise).

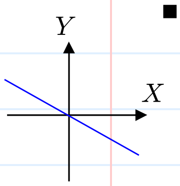




- Theorem. If $X_1, \ldots, X_n$ are independent, and $X_i \sim$ Normal$(\mu_i, \sigma_i^2)$, $i=1, \ldots, n$, then $X_1 + \cdots + X_n \sim$ Normal$(\mu_1 + \cdots + \mu_n, \sigma_1^2 + \cdots + \sigma_n^2)$.

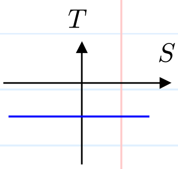
  Proof. (exercise).

- Example. $X$ and $Y$ are random variables with joint pdf $f(x, y)$. Find the distribution of $Z=Y/X$.




- Let

$$Q_z = \{(x,y) : y/x \leq z\}$$
$$= \{(x,y) : x < 0, y \geq zx\} \cup \{(x,y) : x > 0, y \leq zx\}$$




then,

$$F_Z(z) = \int\int_{Q_z} f(x,y)\, dxdy$$
$$= \int_{-\infty}^0 \int_{xz}^\infty + \int_0^\infty \int_{-\infty}^{xz} f(x,y)\, dydx \quad \left(\text{set } \begin{cases} x = s \\ y = st \end{cases}\right)$$
$$= \int_{-\infty}^0 \int_{-\infty}^z + \int_0^\infty \int_{-\infty}^z f(s, st)|s|\, dtds$$
$$= \int_{-\infty}^\infty \int_{-\infty}^z |s| f(s, st)\, dtds$$
$$= \int_{-\infty}^z \int_{-\infty}^\infty |s| f(s, st)\, dsdt$$

and, $f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} |x| f(x, zx)\, dx$

- When $X$ and $Y$ are independent,

$$f(x, y) = f_X(x) f_Y(y).$$

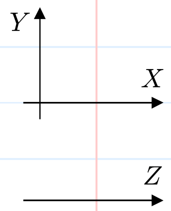
So, $F_Z(z) = \int_{-\infty}^{z} \int_{-\infty}^{\infty} |s| f_X(s) f_Y(st)\, ds dt$

and, $f_Z(z) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(zx)\, dx$

- (exercise) $Z=XY$
- If $X$ and $Y$ are independent,

$$X \sim \text{exponential}(\lambda_1), \text{ and } Y \sim \text{exponential}(\lambda_2),$$

Let $Z=Y/X$. The pdf of $Z$ is




$$\begin{aligned} f_Z(z) &= \int_0^{\infty} x \left(\lambda_1 e^{-\lambda_1 x}\right) \left[\lambda_2 e^{-\lambda_2 (xz)}\right]\, dx \\ &= \frac{\lambda_1 \lambda_2 \Gamma(2)}{(\lambda_1 + \lambda_2 z)^2} \int_0^{\infty} \frac{(\lambda_1 + \lambda_2 z)^2}{\Gamma(2)} x^{2-1} e^{-(\lambda_1 + \lambda_2 z)x}\, dx \\ &= \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2 z)^2} \end{aligned}$$

for $z \geq 0$, and $0$ for $z < 0$.





And, the cdf of $Z$ is

$$\begin{aligned} F_Z(z) &= \int_0^z f_Z(t)\, dt = \int_0^z \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2 t)^2}\, dt \\ &= -\frac{\lambda_1 \lambda_2}{\lambda_2} (\lambda_1 + \lambda_2 t)^{-1} \Big|_0^z = 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2 z} \end{aligned}$$

for $z \geq 0$, and $0$ for $z < 0$.

- Method of probability density function
  - Theorem. Let $\mathbf{X}=(X_1, \ldots, X_n)$ be continuous random variables with the joint pdf $f_{\mathbf{X}}(x_1, \ldots, x_n)$. Let

$$\mathbf{Y}=(Y_1, \ldots, Y_n)= g(\mathbf{X}),$$

where $g$ is 1-to-1, so that its inverse exists and is denoted by

$$\mathbf{x}=g^{-1}(\mathbf{y}) = w(\mathbf{y}) = (w_1(\mathbf{y}), w_2(\mathbf{y}), \ldots, w_n(\mathbf{y})).$$

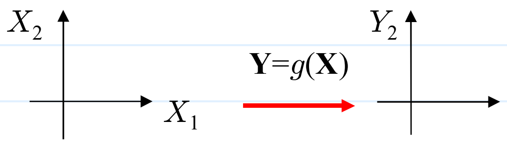
Assume $w$ have continuous partial derivatives. Let

$$J = \begin{vmatrix} \frac{\partial w_1(\mathbf{y})}{\partial y_1} & \frac{\partial w_1(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_1(\mathbf{y})}{\partial y_n} \\ \frac{\partial w_2(\mathbf{y})}{\partial y_1} & \frac{\partial w_2(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_2(\mathbf{y})}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial w_n(\mathbf{y})}{\partial y_1} & \frac{\partial w_n(\mathbf{y})}{\partial y_2} & \cdots & \frac{\partial w_n(\mathbf{y})}{\partial y_n} \end{vmatrix}_{n \times n}$$

Then $\quad f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(g^{-1}(\mathbf{y})) \times |J|,$

for $\mathbf{y}$ s.t. $\mathbf{y}=g(\mathbf{x})$ for some $\mathbf{x}$, and $f_{\mathbf{Y}}(\mathbf{y})=0$, otherwise.

(**Q**: What is the role of $|J|$?)




Proof. $F_{\mathbf{Y}}(y_1, \ldots, y_n) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} f_{\mathbf{Y}}(t_1, \ldots, t_n)\, dt_n \cdots dt_1$

$$= \int \cdots \int_{\substack{(x_1,\ldots,x_n):\\ g_1(x_1,\ldots,x_n)\le y_1\\ \cdots\\ g_n(x_1,\ldots,x_n)\le y_n}} f_{\mathbf{X}}(x_1, \ldots, x_n)\, dx_n \cdots dx_1.$$

It then follows from an exercise in advanced calculus that

$$f_{\mathbf{Y}}(y_1, \ldots, y_n) = \frac{\partial^n}{\partial y_1 \cdots \partial y_n} F_{\mathbf{Y}}(y_1, \ldots, y_n)$$
$$= f_{\mathbf{X}}(w_1(\mathbf{y}), \ldots, w_n(\mathbf{y})) \times |J|.$$

- Remark. When the dimensionality of $\mathbf{Y}$ (denoted by $k$) is less than $n$, we can choose another $n-k$ transformations $\mathbf{Z}$ such that

$$(\mathbf{Y}, \mathbf{Z})=g(\mathbf{X})$$

satisfy the assumptions in above theorem.





By integrating out the last $n-k$ arguments in the joint pdf of $(\mathbf{Y}, \mathbf{Z})$, the joint pdf of $\mathbf{Y}$ can be obtained.

- Example. $X_1$ and $X_2$ are random variables with joint pdf $f_{\mathbf{X}}(x_1, x_2)$. Find the distribution of $Y_1=X_1/(X_1+X_2)$.
  - Let $Y_2=X_1+X_2$, then

$$x_1 = y_1 y_2 \equiv w_1(y_1, y_2)$$
$$x_2 = y_2 - y_1 y_2 \equiv w_2(y_1, y_2).$$

Since $\frac{\partial w_1}{\partial y_1} = y_2, \quad \frac{\partial w_1}{\partial y_2} = y_1, \quad \frac{\partial w_2}{\partial y_1} = -y_2, \quad \frac{\partial w_2}{\partial y_2} = 1 - y_1,$

$$J = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} = y_2 - y_1 y_2 + y_1 y_2 = y_2, \text{ and } |J| = |y_2|.$$

Therefore, $f_{\mathbf{Y}}(y_1, y_2) = f_{\mathbf{X}}(y_1 y_2, y_2 - y_1 y_2)|y_2|,$

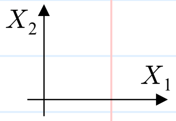
and, $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{\mathbf{Y}}(y_1, y_2)\, dy_2$

$$= \int_{-\infty}^{\infty} f_{\mathbf{X}}(y_1 y_2, y_2 - y_1 y_2)|y_2|\, dy_2.$$

$$\left(= \int_{-\infty}^{\infty} f_{X_1}(y_1 y_2) f_{X_2}(y_2 - y_1 y_2)|y_2|\, dy_2\right.$$

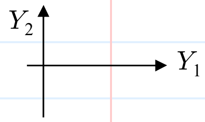
when $X_1$ and $X_2$ are independent)

- Theorem. If $X_1$ and $X_2$ are independent, and




$$X_1 \sim \text{Gamma}(\alpha_1, \lambda), \quad X_2 \sim \text{Gamma}(\alpha_2, \lambda),$$

then $Y_1 = X_1/(X_1+X_2) \sim \text{Beta}(\alpha_1, \alpha_2)$.




Proof. For $x_1, x_2 \geq 0$, the joint pdf of $\mathbf{X}$ is

$$\begin{aligned} f_{\mathbf{X}}(x_1, x_2) &= \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} x_1^{\alpha_1 - 1} e^{-\lambda x_1} \times \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} x_2^{\alpha_2 - 1} e^{-\lambda x_2} \\ &= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x_1^{\alpha_1-1} x_2^{\alpha_2-1} e^{-\lambda(x_1+x_2)}. \end{aligned}$$

So, for $0 \leq y_1 \leq 1$,

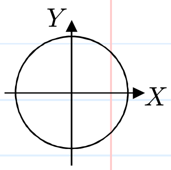
$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f_{X_1}(y_1 y_2) f_{X_2}(y_2 - y_1 y_2) |y_2| \, dy_2 \\ &= \int_0^\infty \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (y_1 y_2)^{\alpha_1-1} (y_2 - y_1 y_2)^{\alpha_2-1} e^{-\lambda y_2} \cdot y_2 \, dy_2 \\ &= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1} \\ &\quad \times \int_0^\infty \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} y_2^{(\alpha_1+\alpha_2)-1} e^{-\lambda y_2} \, dy_2 \\ &= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1} \end{aligned}$$

and $f_{Y_1}(y_1) = 0$, otherwise.

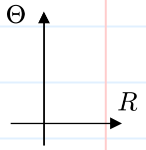




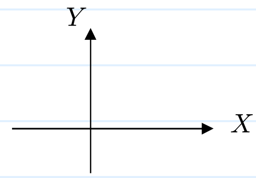
- Example. Suppose that $X$ and $Y$ have a uniform distribution over the region $D = \{(x, y): x^2+y^2 \leq 1\}$, i.e., their joint pdf is




$$f_{X,Y}(x, y) = \frac{1}{\pi} \mathbf{1}_D(x, y).$$




Find the joint distribution of $(R, \Theta)$ and examine whether $R$ and $\Theta$ are independent, where $(R, \Theta)$ is the polar coordinate representation of $(X, Y)$, i.e.,




$$\begin{aligned} X &= R\cos(\Theta) \equiv w_1(R, \Theta), \\ Y &= R\sin(\Theta) \equiv w_2(R, \Theta). \end{aligned}$$

- Since $\dfrac{\partial w_1}{\partial r} = \cos(\theta)$, $\quad \dfrac{\partial w_1}{\partial \theta} = -r\sin(\theta)$,
  $\dfrac{\partial w_2}{\partial r} = \sin(\theta)$, $\quad \dfrac{\partial w_2}{\partial \theta} = r\cos(\theta)$,

$$J = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r,$$

  and $|J| = |r| = r$.
- For $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$, the joint pdf of $(R, \Theta)$ is

$$f_{R,\Theta}(r, \theta) = f_{X,Y}(r\cos(\theta), r\sin(\theta)) \times |J| = \frac{1}{\pi} r$$

  and $f_{R,\Theta}(r, \theta) = 0$, otherwise.

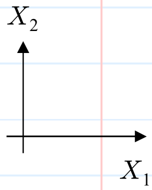
- By the theorem in LNp.7-25, $(R, \Theta)$ are independent.

- Example. Let $X_1, \ldots, X_n$ be **independent** and **identically distributed** (i.e., **i.i.d.**) exponential($\lambda$). Let

$$Y_i = X_1 + \cdots + X_i,\ i = 1, \ldots, n.$$

Find the distribution of $\mathbf{Y}=(Y_1, \ldots, Y_n)$.

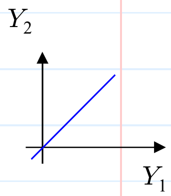
[Note. It has been shown that $Y_i \sim$ Gamma($i$, $\lambda$), $i$=1, …, $n$.]




- The joint pdf of $X_1, \ldots, X_n$ is

$$f_{\mathbf{X}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) = \prod_{i=1}^{n} \left(\lambda e^{-\lambda x_i}\right) = \lambda^n e^{-\lambda(x_1+\cdots+x_n)}.$$

for $0 \le x_i < \infty$, $i$=1, …, $n$.




- Since

$$\begin{aligned} x_1 &= y_1 &&\equiv w_1(y_1, \ldots, y_n),\\ x_2 &= y_2 - y_1 &&\equiv w_2(y_1, \ldots, y_n),\\ &\cdots \\ x_n &= y_n - y_{n-1} &&\equiv w_n(y_1, \ldots, y_n), \end{aligned}$$

we have

$$\frac{\partial w_i}{\partial y_j} = \begin{cases} 1, & \text{if } j = i,\\ -1, & \text{if } j = i-1,\\ 0, & \text{otherwise,} \end{cases}$$





$$J = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0\\ -1 & 1 & 0 & \cdots & 0\\ 0 & -1 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1, \text{ and } |J| = 1.$$

- For $0 \le y_1 \le y_2 \le \cdots \le y_{i-1} \le y_i \le y_{i+1} \le \cdots \le y_n < \infty$,

$$\begin{aligned} f_{\mathbf{Y}}(y_1, \ldots, y_n) &= f_{\mathbf{X}}(y_1, y_2 - y_1, \ldots, y_n - y_{n-1}) \times |J| \\ &= \lambda^n e^{-\lambda y_n}. \end{aligned}$$

and $f_{\mathbf{Y}}(y_1, \ldots, y_n) = 0$, otherwise.

- The marginal pdf of $Y_i$ is

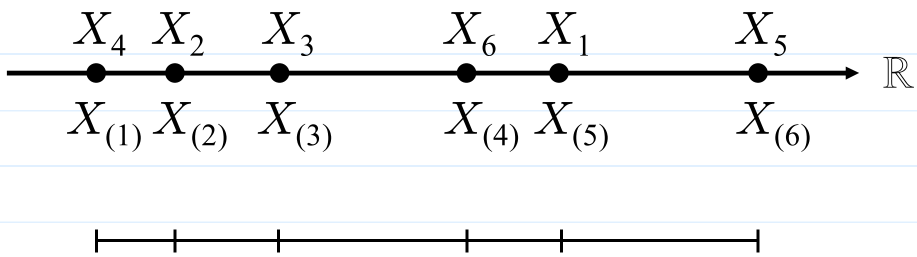
$$\begin{aligned} f_{Y_i}(y) &= \int_0^y \int_{y_1}^y \cdots \int_{y_{i-2}}^y \int_y^\infty \int_{y_{i+1}}^\infty \cdots \int_{y_{n-1}}^\infty \\ &\qquad \lambda^n e^{-\lambda y_n}\, dy_n \cdots dy_{y_{i+2}} dy_{i+1} dy_{i-1} \cdots dy_2 dy_1 \\ &= \int_0^y \int_{y_1}^y \cdots \int_{y_{i-2}}^y \lambda^i e^{-\lambda y}\, dy_{i-1} \cdots dy_2 dy_1 \\ &= \lambda^i e^{-\lambda y} \frac{y^{i-1}}{(i-1)!}, \end{aligned}$$

for $y \ge 0$, and $f_{Y_i}(y) = 0$, otherwise.

➢ Method of moment generating function.

- Based on the *uniqueness theorem* of moment generating function to be explained later in Chapter 7
- Especially useful to identify the distribution of sum of independent random variables.

• Order Statistics




➢ Definition. Let $X_1, \ldots, X_n$ be random variables. We sort the $X_i$'s and denote by

$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$$

the *order statistics*. Using the notation,





$X_{(i)}$ = $i$th-smallest value in $X_1, \ldots, X_n$, $i$=1, 2, …, $n$,

$X_{(1)} = \min(X_1, \ldots, X_n)$ is the *minimum*,

$X_{(n)} = \max(X_1, \ldots, X_n)$ is the *maximum*,

$R \equiv X_{(n)} - X_{(1)}$ is called *range*,

$S_j \equiv X_{(j)} - X_{(j-1)}$, $j$=2, …, $n$, are called *$j$th spacing*.

**Q**: What are the joint distributions of various order statistics and their marginal distributions?

➢ Definition. $X_1, \ldots, X_n$ are called *i.i.d.* (**i**ndependent, **i**dentically **d**istributed) with cdf $F$/pdf $f$/pmf $p$ if the random variables $X_1, \ldots, X_n$ are independent and have a common marginal distribution with cdf $F$/pdf $f$/pmf $p$.

- Remark. In the discussion about order statistics, we only consider the case that $X_1, \ldots, X_n$ are i.i.d.
  - Note. Although $X_1, \ldots, X_n$ are independent, their order statistics $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ are *not* independent in general.

➢ Theorem. Suppose that $X_1, \ldots, X_n$ are i.i.d. with cdf $F$.

1. The cdf of $X_{(1)}$ is $1-[1-F(x)]^n$, and the cdf of $X_{(n)}$ is $[F(x)]^n$.
2. If $\mathbf{X}$ are continuous and $F$ has a pdf $f$, then the pdf of $X_{(1)}$ is $nf(x)[1-F(x)]^{n-1}$, and the pdf of $X_{(n)}$ is $nf(x)[F(x)]^{n-1}$.

Proof. By the method of cumulative distribution function,

$$\begin{aligned} 1 - F_{X_{(1)}}(x) &= P(X_{(1)} > x) = P(X_1 > x, \ldots, X_n > x) \\ &= P(X_1 > x) \cdots P(X_n > x) = [1 - F(x)]^n. \end{aligned}$$

$$\begin{aligned} F_{X_{(n)}}(x) &= P(X_{(n)} \le x) = P(X_1 \le x, \ldots, X_n \le x) \\ &= P(X_1 \le x) \cdots P(X_n \le x) = [F(x)]^n. \end{aligned}$$

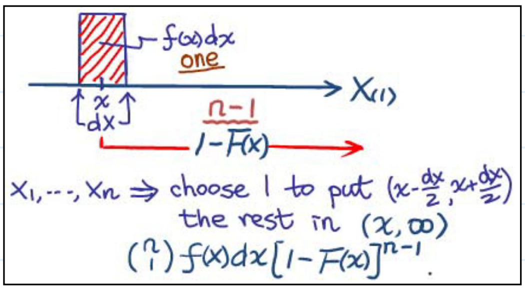
$$\begin{aligned} f_{X_{(1)}}(x) &= \tfrac{d}{dx} F_{X_{(1)}}(x) \\ &= n[1 - F(x)]^{n-1} \left(\tfrac{d}{dx} F(x)\right) = nf(x)[1 - F(x)]^{n-1}. \end{aligned}$$

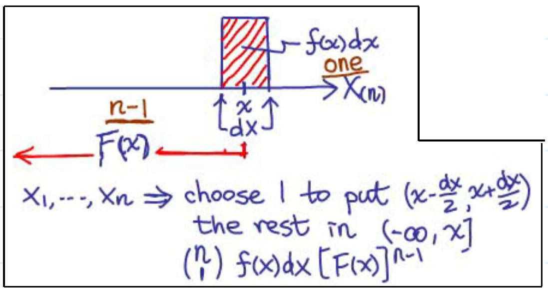
$$\begin{aligned} f_{X_{(n)}}(x) &= \tfrac{d}{dx} F_{X_{(n)}}(x) \\ &= n[F(x)]^{n-1} \left(\tfrac{d}{dx} F(x)\right) = nf(x)[F(x)]^{n-1}. \end{aligned}$$





- Graphical interpretation for the pdfs of $X_{(1)}$ and $X_{(n)}$.




- Example. $n$ light bulbs are placed in service at time $t=0$, and allowed to burn continuously. Denote their lifetimes by $X_1, \ldots, X_n$, and suppose that they are i.i.d. with cdf $F$.

  If burned out bulbs are not replaced, then the room goes dark at time
  $$Y = X_{(n)} = \max(X_1, \ldots, X_n).$$

  - If $n=5$ and $F$ is exponential with $\lambda = 1$ per month, then
  $$F(x) = 1-e^{-x}, \text{ for } x \ge 0, \text{ and } 0, \text{ for } x < 0.$$
  - The cdf of $Y$ is
  $$F_Y(y) = (1-e^{-y})^5, \text{ for } y \ge 0, \text{ and } 0, \text{ for } y < 0,$$
  and its pdf is $5(1-e^{-y})^4 e^{-y}$, for $y \ge 0$, and 0, for $y < 0$.

- The probability that the room is still lighted after two months is $P(Y>2) = 1-F_Y(2) = 1-(1-e^{-2})^5$.

➢ Theorem. Suppose that $X_1, \ldots, X_n$ are i.i.d. with pmf $p$/pdf $f$. Then, the joint pmf/pdf of $X_{(1)}, \ldots, X_{(n)}$ is

$$p_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n) = n! \times p(x_1) \times \cdots \times p(x_n),$$

or

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n) = n! \times f(x_1) \times \cdots \times f(x_n),$$

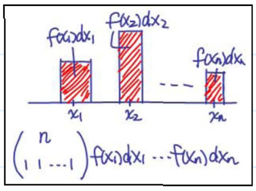
for $x_1 \le x_2 \le \cdots \le x_n$, and 0 otherwise.

Proof. For $x_1 \le x_2 \le \cdots \le x_n$,

$$\begin{aligned} p_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n) &= P(X_{(1)} = x_1, \ldots, X_{(n)} = x_n) \\ &= \sum_{\substack{(i_1,\ldots,i_n):\\ \text{permutations of}\\ (1,\ldots,n)}} P(X_1 = x_{i_1}, \ldots, X_n = x_{i_n}) \\ &= \sum_{\substack{(i_1,\ldots,i_n):\\ \text{permutations of}\\ (1,\ldots,n)}} p(x_1) \times \cdots \times p(x_n) \\ &= n! \times p(x_1) \times \cdots \times p(x_n). \end{aligned}$$

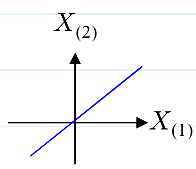







$$\begin{aligned} &f_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n)\, dx_1 \cdots dx_n \\ &\approx P\left(x_1 - \tfrac{dx_1}{2} < X_{(1)} < x_1 + \tfrac{dx_1}{2}, \ldots, x_n - \tfrac{dx_n}{2} < X_{(n)} < x_n + \tfrac{dx_n}{2}\right) \\ &= \sum_{\substack{(i_1,\ldots,i_n):\\ \text{permutations of}\\ (1,\ldots,n)}} P\left(x_{i_1} - \tfrac{dx_{i_1}}{2} < X_1 < x_{i_1} + \tfrac{dx_{i_1}}{2}, \ldots, x_{i_n} - \tfrac{dx_{i_n}}{2} < X_n < x_{i_n} + \tfrac{dx_{i_n}}{2}\right) \\ &\approx \sum_{\substack{(i_1,\ldots,i_n):\\ \text{permutations of}\\ (1,\ldots,n)}} f(x_1) \times \cdots \times f(x_n)\, dx_1 \cdots dx_n \\ &= n! \times f(x_1) \times \cdots \times f(x_n)\, dx_1 \cdots dx_n. \end{aligned}$$

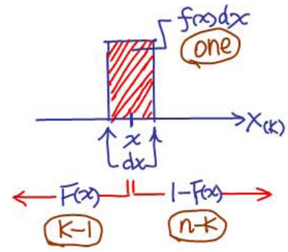
- **Q**: Examine whether $X_{(1)}, \ldots, X_{(n)}$ are independent using the Theorem in LNp.7-25.




➢ Theorem. If $X_1, \ldots, X_n$ are i.i.d. with cdf $F$ and pdf $f$, then

1. The pdf of the $k^{\text{th}}$ order statistic $X_{(k)}$ is
$$f_{X_{(k)}}(x) = \binom{n}{1,k-1,n-k} f(x) F(x)^{k-1} [1-F(x)]^{n-k}.$$
2. The cdf of $X_{(k)}$ is
$$F_{X_{(k)}}(x) = \sum_{m=k}^{n} \binom{n}{m} [F(x)]^m [1-F(x)]^{n-m}.$$

Proof.




$X_1, \ldots, X_n \Rightarrow$ choose 1 to place in $(x-\frac{dx}{2}, x+\frac{dx}{2})$

" $k-1$ " " " $(-\infty, x)$

" $n-k$ " " " $(x, \infty)$

$\binom{n}{1\;\; k-1\;\; n-k} f(x)dx\,[F(x)]^{k-1}[1-F(x)]^{n-k}$

$$F_{X_{(k)}}(x) = P(X_{(k)} \le x)$$
$$= P(\text{at least } k \text{ of the } X_i\text{'s are} \le x)$$
$$= \sum_{m=k}^{n} P(\text{exact } m \text{ of the } X_i\text{'s are} \le x) \longrightarrow X_{(k)}$$
$$= \sum_{m=k}^{n} \binom{n}{m} [F(x)]^m [1-F(x)]^{n-m}$$

➢ Theorem. If $X_1, \ldots, X_n$ are i.i.d. with cdf $F$ and pdf $f$, then

1. The joint pdf of $X_{(1)}$ and $X_{(n)}$ is

$$f_{X_{(1)},X_{(n)}}(s,t) = n(n-1)f(s)f(t)[F(t)-F(s)]^{n-2},$$

for $s \le t$, and 0 otherwise.

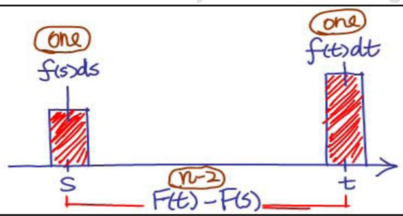
2. The pdf of the range $R = X_{(n)} - X_{(1)}$ is

$$f_R(r) = \int_{-\infty}^{\infty} f_{X_{(1)},X_{(n)}}(u, u+r)\, du,$$

for $r \ge 0$, and 0 otherwise.








$X_1, \ldots, X_n \Rightarrow$ choose one to place in $(s-\frac{ds}{2}, s+\frac{ds}{2})$

one $(t-\frac{dt}{2}, t+\frac{dt}{2})$

the rest in $(s,t)$

$\binom{n}{1\;\; 1\;\; n-2} f(s)ds\,f(t)dt\,[F(t)-F(s)]^{n-2}$

➢ Theorem. If $X_1, \ldots, X_n$ are i.i.d. with cdf $F$ and pdf $f$, then

1. The joint pdf of $X_{(i)}$ and $X_{(j)}$, where $1 \le i < j \le n$, is

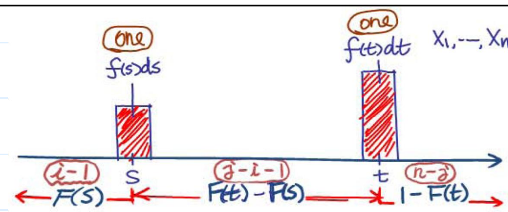
$$f_{X_{(i)},X_{(j)}}(s,t) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} f(s)f(t) \times [F(s)]^{i-1}[F(t)-F(s)]^{j-i-1}[1-F(t)]^{n-j},$$

for $s \le t$, and 0 otherwise.

2. The pdf of the $j^{\text{th}}$ spacing $S_j = X_{(j)} - X_{(j-1)}$ is

$$f_{S_j}(s) = \int_{-\infty}^{\infty} f_{X_{(j-1)},X_{(j)}}(u, u+s)\, du,$$

for s $\ge 0$, and zero otherwise.




$X_1, \ldots, X_n \Rightarrow$ choose one to place in $(s-\frac{ds}{2}, s+\frac{ds}{2})$

one - - - $(t-\frac{dt}{2}, t+\frac{dt}{2})$

$i-1$ - - - $(-\infty, s)$

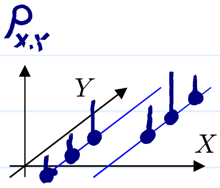
$j-i-1$ - - - $(s, t)$

the rest - - - $(t, \infty)$

$\binom{n}{1,1,i-1,j-i-1,n-j} f(s)ds\,f(t)\,dt \times [F(s)]^{i-1}[F(t)-F(s)]^{j-i-1}[1-F(t)]^{n-j}$

❖ **Reading**: textbook, Sec 6.3, 6.6, 6.7

# Conditional Distribution




- Definition. Let $\mathbf{X}$ $(\in \mathbb{R}^n)$ and $\mathbf{Y}$ $(\in \mathbb{R}^m)$ be discrete random vectors and $(\mathbf{X}, \mathbf{Y})$ have a joint pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$, then the *conditional joint pmf* of $\mathbf{Y}$ given $\mathbf{X}=\mathbf{x}$ is defined as

$$p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \equiv P(\{\mathbf{Y}=\mathbf{y}\}|\{\mathbf{X}=\mathbf{x}\}) = \frac{P(\{\mathbf{X}=\mathbf{x}, \mathbf{Y}=\mathbf{y}\})}{P(\{\mathbf{X}=\mathbf{x}\})} = \frac{p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{p_{\mathbf{X}}(\mathbf{x})} = \frac{\text{joint}}{\text{marginal}}$$

if $p_{\mathbf{X}}(\mathbf{x})>0$. The probability is defined to be zero if $p_{\mathbf{X}}(\mathbf{x})=0$.

➢ Some Notes.

- For each fixed $\mathbf{x}$, $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ is a joint pmf for $\mathbf{y}$, since

$$\sum_{\mathbf{y}} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = \frac{1}{p_{\mathbf{X}}(\mathbf{x})} \sum_{\mathbf{y}} p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = \frac{1}{p_{\mathbf{X}}(\mathbf{x})} \times p_{\mathbf{X}}(\mathbf{x}) = 1.$$

- For an event $B$ of $\mathbf{Y}$, the probability that $\mathbf{Y}\in B$ given $\mathbf{X}=\mathbf{x}$ is

$$P(\mathbf{Y} \in B | \mathbf{X}=\mathbf{x}) = \sum_{\mathbf{u}\in B} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{u}|\mathbf{x}).$$





- The *conditional joint cdf* of $\mathbf{Y}$ given $\mathbf{X}=\mathbf{x}$ can be similarly defined from the conditional joint pmf $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$, i.e.,

$$F_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = P(\mathbf{Y} \le \mathbf{y}|\mathbf{X}=\mathbf{x}) = \sum_{\mathbf{u}\le\mathbf{y}} p_{\mathbf{Y}|\mathbf{X}}(\mathbf{u}|\mathbf{x}).$$

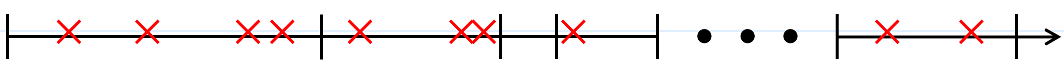
➢ Theorem. Let $X_1, \ldots, X_m$ be independent and

$$X_i \sim \text{Poisson}(\lambda_i),\ i=1, \ldots, m.$$

Let $Y=X_1+\cdots+X_m$, then

$$(X_1, \ldots, X_m | Y=n) \sim \text{Multinomial}(n, m, p_1, \ldots, p_m),$$

where $p_i = \lambda_i/(\lambda_1+\cdots+\lambda_m)$ for $i=1, \ldots, m$.



Proof. The joint pmf of $(X_1, \ldots, X_m, Y)$ is

$$p_{\mathbf{X},Y}(x_1, \ldots, x_m, n) = P(\{X_1=x_1, \ldots, X_m=x_m\} \cap \{Y=n\})$$
$$= \begin{cases} P(X_1=x_1, \ldots, X_m=x_m), & \text{if } x_1+\cdots+x_m=n, \\ 0, & \text{if } x_1+\cdots+x_m \neq n. \end{cases}$$

Furthermore, the distribution of $Y$ is Poisson($\lambda_1+\cdots+\lambda_m$), i.e.,

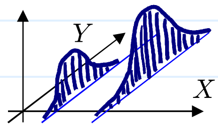
$$p_Y(n) = P(Y = n) = \frac{e^{-(\lambda_1+\cdots+\lambda_m)}(\lambda_1+\cdots+\lambda_m)^n}{n!}.$$

Therefore, for $\mathbf{x}=(x_1, \ldots, x_m)$ wheres $x_i \in \{0,1,2, \ldots\}$, $i=1, \ldots, m$, and $x_1+\cdots+x_m=n$, the conditional joint pmf of $\mathbf{X}$ given $Y=n$ is

$$p_{\mathbf{X}|Y}(\mathbf{x}|n) = \frac{p_{\mathbf{X},Y}(x_1,\ldots,x_m,n)}{p_Y(n)} = \frac{\prod_{i=1}^{m} \frac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}}{\frac{e^{-(\lambda_1+\cdots+\lambda_m)}(\lambda_1+\cdots+\lambda_m)^n}{n!}}$$

$$= \frac{n!}{x_1!\times\cdots\times x_m!} \times \left(\frac{\lambda_1}{\lambda_1+\cdots+\lambda_m}\right)^{x_1} \times\cdots\times \left(\frac{\lambda_m}{\lambda_1+\cdots+\lambda_m}\right)^{x_m}.$$

- Definition. Let $\mathbf{X}$ ($\in\mathbb{R}^n$) and $\mathbf{Y}$ ($\in\mathbb{R}^m$) be continuous random vectors and $(\mathbf{X}, \mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$, then the *conditional joint pdf* of $\mathbf{Y}$ given $\mathbf{X}=\mathbf{x}$ is defined as




$$f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \equiv \frac{f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{f_{\mathbf{X}}(\mathbf{x})} = \frac{\text{joint}}{\text{marginal}},$$

if $f_{\mathbf{X}}(\mathbf{x})>0$, and 0 otherwise.





➢ Some Notes.

- $P(\mathbf{X}=\mathbf{x})=0$ for a continuous random vector $\mathbf{X}$.
- The justification of $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ comes from

$$P(\mathbf{Y} \le \mathbf{y} | \mathbf{x} - (\Delta\mathbf{x}/2) < \mathbf{X} \le \mathbf{x} + (\Delta\mathbf{x}/2))$$

$$= \frac{\int_{-\infty}^{\mathbf{y}} \int_{\mathbf{x}-(\Delta\mathbf{x}/2)}^{\mathbf{x}+(\Delta\mathbf{x}/2)} f_{\mathbf{X},\mathbf{Y}}(\mathbf{u},\mathbf{v})\, d\mathbf{u}d\mathbf{v}}{\int_{\mathbf{x}-(\Delta\mathbf{x}/2)}^{\mathbf{x}+(\Delta\mathbf{x}/2)} f_{\mathbf{X}}(\mathbf{t})\, d\mathbf{t}}$$

$$\approx \frac{\int_{-\infty}^{\mathbf{y}} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{v})|\Delta\mathbf{x}|\, d\mathbf{v}}{f_{\mathbf{X}}(\mathbf{x})|\Delta\mathbf{x}|} = \int_{-\infty}^{\mathbf{y}} \frac{f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{v})}{f_{\mathbf{X}}(\mathbf{x})}\, d\mathbf{v}$$

- For each fixed $\mathbf{x}$, $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ is a joint pdf for $\mathbf{y}$, since

$$\int_{-\infty}^{\infty} f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y} = \frac{1}{f_{\mathbf{X}}(\mathbf{x})} \int_{-\infty}^{\infty} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})\, d\mathbf{y} = \frac{1}{f_{\mathbf{X}}(\mathbf{x})} \times f_{\mathbf{X}}(\mathbf{x}) = 1.$$

- For an event $B$ of $\mathbf{Y}$, we can write

$$P(\mathbf{Y} \in B | \mathbf{X} = \mathbf{x}) = \int_B f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\, d\mathbf{y}.$$

- The *conditional joint cdf* of $\mathbf{Y}$ given $\mathbf{X}=\mathbf{x}$ can be similarly defined from the conditional joint pdf $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$, i.e.,

$$F_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = P(\mathbf{Y} \le \mathbf{y} | \mathbf{X} = \mathbf{x}) = \int_{-\infty}^{\mathbf{y}} f_{\mathbf{Y}|\mathbf{X}}(\mathbf{t}|\mathbf{x})\, d\mathbf{t}.$$

➢ Example. If $X$ and $Y$ have a joint pdf

$$f(x,y)=\frac{2}{(1+x+y)^3},$$

for $0\le x,\ y<\infty$, then

$$f_X(x)=\int_0^\infty f(x,y)\,dy=-\left.\frac{1}{(1+x+y)^2}\right|_0^\infty=\frac{1}{(1+x)^2},$$

for $0\le x<\infty$. So, $$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\frac{2(1+x)^2}{(1+x+y)^3},$$

and,
$$\begin{aligned} P(Y>c|X=x) &= \int_c^\infty \frac{2(1+x)^2}{(1+x+y)^3}\,dy \\ &= -\left.\frac{(1+x)^2}{(1+x+y)^2}\right|_{y=c}^\infty=\frac{(1+x)^2}{(1+x+c)^2}. \end{aligned}$$

- Mixed Joint Distribution: Definition of conditional distribution can be similarly generalized to the case in which some random variables are discrete and the others continuous (see a later example).

- Theorem (Multiplication Law). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X},\mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$, then

$$p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})=p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\times p_{\mathbf{X}}(\mathbf{x}),\quad \text{or}$$
$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})=f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\times f_{\mathbf{X}}(\mathbf{x}).$$

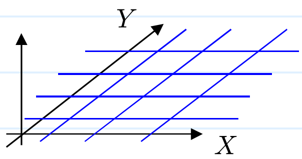




Proof. By the definition of conditional distribution.

- Theorem (Law of Total Probability). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X},\mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$, then

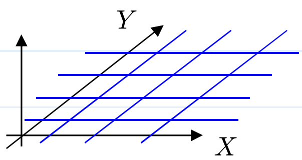
$$p_{\mathbf{Y}}(\mathbf{y})=\sum_{\mathbf{x}=-\infty}^{\infty}p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})p_{\mathbf{X}}(\mathbf{x}),\quad \text{or}$$
$$f_{\mathbf{Y}}(\mathbf{y})=\int_{-\infty}^{\infty}f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x}.$$




Proof. By the definition of marginal distribution and the multiplication law.

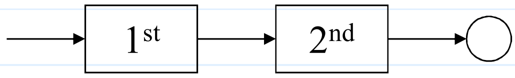
- Theorem (Bayes Theorem). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X},\mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})$, then

$$p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})=\frac{p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})p_{\mathbf{X}}(\mathbf{x})}{\sum_{\mathbf{x}=-\infty}^{\infty}p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})p_{\mathbf{X}}(\mathbf{x})},\quad \text{or}$$
$$f_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})=\frac{f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})f_{\mathbf{X}}(\mathbf{x})}{\int_{-\infty}^{\infty}f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x}}.$$




Proof. By the definition of conditional distribution, multiplication law, and the law of total probability.

➢ Example.




(LNp.4-14)

- Suppose that $X \sim$ Uniform(0, 1), and $(Y_1, \ldots, Y_n | X=x)$ are i.i.d. with Bernoulli($x$), i.e.,

$$p_{\mathbf{Y}|X}(y_1, \ldots, y_n | x) = x^{y_1+\cdots+y_n}(1-x)^{n-(y_1+\cdots+y_n)},$$

for $y_1, \ldots, y_n \in \{0, 1\}$.

- By the multiplication law, for $y_1, \ldots, y_n \in \{0, 1\}$ and $0<x<1$,

$$p_{\mathbf{Y},X}(y_1, \ldots, y_n, x) = x^{y_1+\cdots+y_n}(1-x)^{n-(y_1+\cdots+y_n)}.$$

- Suppose that we observed $Y_1=1, \ldots, Y_n=1$.
- By the law of total probability,

$$\begin{aligned} P(Y_1 = 1, \ldots, Y_n = 1) &= p_{\mathbf{Y}}(1, \ldots, 1) \\ &= \int_0^1 p_{\mathbf{Y}|X}(1, \ldots, 1|x) f_X(x)\, dx \\ &= \int_0^1 x^n\, dx = \frac{1}{n+1} x^{n+1}\Big|_0^1 = \frac{1}{n+1}. \end{aligned}$$





- And, by Bayes' Theorem,

$$\begin{aligned} f_{X|\mathbf{Y}}(x|Y_1 = 1, \ldots, Y_n = 1) \\ = \frac{p_{\mathbf{Y}|X}(1, \ldots, 1|x) f_X(x)}{p_{\mathbf{Y}}(1, \ldots, 1)} = (n+1)x^n. \end{aligned}$$

for $0<x<1$, i.e., $(X|Y_1=1, \ldots, Y_n=1) \sim$ Beta($n$+1, 1).

(cf., marginal distribution of $X \sim$ Uniform(0, 1)=Beta(1, 1).)

- If there were an $(n+1)^{\text{st}}$ Bernoulli trial $Y_{n+1}$,

$$\begin{aligned} P(Y_{n+1} = 1|Y_1 = 1, \ldots, Y_n = 1) \\ = \frac{P(Y_1 = 1, \ldots, Y_{n+1} = 1)}{P(Y_1 = 1, \ldots, Y_n = 1)} = \frac{1/(n+2)}{1/(n+1)} = \frac{n+1}{n+2}. \end{aligned}$$

- (exercise) In general, it can be shown that

$(X|Y_1=y_1, \ldots, Y_n=y_n) \sim$ Beta$((y_1+\cdots+y_n)+1, n-(y_1+\cdots+y_n)+1)$.

- Theorem (Conditional Distribution & Independent). Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors and $(\mathbf{X}, \mathbf{Y})$ have a joint pdf $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$/pmf $p_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, \mathbf{y})$. Then, $\mathbf{X}$ and $\mathbf{Y}$ are independent, i.e.,

$$p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = p_{\mathbf{X}}(\mathbf{x}) \times p_{\mathbf{Y}}(\mathbf{y}), \quad \text{or}$$
$$f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \times f_{\mathbf{Y}}(\mathbf{y}),$$

if and only if

$$p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = p_{\mathbf{Y}}(\mathbf{y}), \quad \text{or}$$
$$f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) = f_{\mathbf{Y}}(\mathbf{y}).$$

Proof. By the definition of conditional distribution.

➢ Intuition.

- The 2 graphs about the joint pmf/pdf of independent r.v.'s in LNp.7-27
- $p_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ or $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ offers information about the distribution of **Y** when **X**=**x**.

  $p_{\mathbf{Y}}(\mathbf{y})$ or $f_{\mathbf{Y}}(\mathbf{y})$ offers information about the distribution of **Y** when **X** not observed.

❖ **Reading**: textbook, Sec 6.4, 6.5