• Beta Distribution 貝他

➢Beta Function:

$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty u^{\alpha-1}e^{-u}du \cdot \int_0^\infty v^{\beta-1}e^{-v}dv = \int_0^\infty\int_0^\infty u^{\alpha-1}v^{\beta-1}e^{-(u+v)}du\,dv$

Let $u = zx$, $v = z(1-x)$ ($u+v=z$)

x, $1-x$: (0) x $1-x$ (1)

binomial coefficient: $\frac{1}{\binom{n}{x}} = \frac{x!(n-x)!}{n!}$

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \xrightarrow{\text{if } \alpha,\beta:\text{ integers}} \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$$

(exercise) check LNp.6-22~23 c.f.

➢For α, β > 0 (fixed constants), the function

binomial pmf: $\binom{n}{x}p^x(1-p)^{n-x}$ c.f.

possible values of X

$$f(x) = \begin{cases} \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{=\frac{1}{B(\alpha,\beta)}} x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

from Beta function

is a pdf (exercise).

▪ The distribution of a random variable X with this pdf is called the *beta* distribution with parameters α and β.

➢The cdf of beta distribution can be expressed in terms of the *incomplete beta function*, i.e., $F(x)=0$ for $x<0$, $F(x)=1$ for $x>1$, and for $0 \le x \le 1$,

α, β: integers: Beta(α, β) ⟷ $P(X \le x) = P(Y \ge \alpha)$ ⟷ binomial $(\alpha+\beta-1, x)$

c.f. binomial & negative binomial in LNp.6-24

$$F(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x y^{\alpha-1}(1-y)^{\beta-1}\,dy \equiv \frac{B(x;\alpha,\beta)}{B(\alpha,\beta)}.$$

incomplete beta function (g', f)

Integration by parts

$$= \text{(exercise)} \sum_{i=\alpha}^{\alpha+\beta-1} \frac{(\alpha+\beta-1)!}{i!(\alpha+\beta-1-i)!}x^i(1-x)^{\alpha+\beta-1-i},$$

for *integer values* of α and β

$$= \sum_{i=\alpha}^{\alpha+\beta-1}\binom{\alpha+\beta-1}{i}x^i(1-x)^{(\alpha+\beta-1)-i}$$

pmf of binomial $(\alpha+\beta-1, x)$

➢Theorem. The mean and variance of a beta distribution with parameters α and β are

Interpretation → $\mu = \frac{\alpha}{\alpha+\beta}$ and $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

$E(X^2)-[E(X)]^2$; $= \frac{\alpha}{\alpha+\beta}\cdot\frac{\beta}{\alpha+\beta}\cdot\frac{1}{\alpha+\beta+1}$

c.f. $Var\left(\frac{X}{n}\right) = \frac{p(1-p)}{n}$, $X \sim$ binomial(n,p)

Proof.

$$E(X) = \int_0^1 x\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}}\cdot\frac{\Gamma(\alpha+1)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+1)}\int_0^1 \underbrace{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)}x^{(\alpha+1)-1}(1-x)^{\beta-1}}_{\text{pdf of Beta}(\alpha+1,\beta)}\,dx$$

$\Gamma(\alpha+1) = \alpha\cancel{\Gamma(\alpha)}$, $\Gamma(\alpha+\beta+1) = (\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}$

$$= \frac{\alpha}{\alpha+\beta}.$$

$$E(X^2) = \int_0^1 x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}} \frac{\Gamma(\alpha+2)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+2)} \underbrace{\int_0^1 \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta)} x^{(\alpha+2)-1}(1-x)^{\beta-1}\,dx}_{\text{pdf of Beta}(\alpha+2,\beta)}$$

$\Gamma(\alpha+2) = (\alpha+1)\alpha\,\cancel{\Gamma(\alpha)}$, $\Gamma(\alpha+\beta+2) = (\alpha+\beta+1)(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}.$$

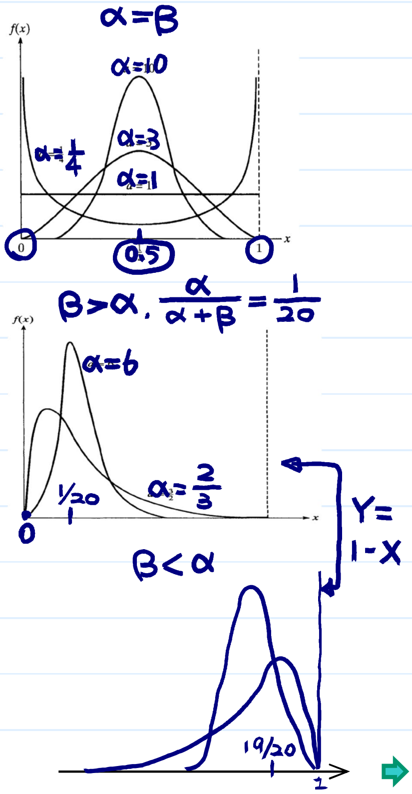
- Some properties
  - When $\alpha=\beta=1$, the beta distribution is the same as the uniform(0, 1).
  - Whenever $\alpha=\beta$, the beta distribution is symmetric about $x=0.5$, i.e.,

    $$f(0.5-\Delta)=f(0.5+\Delta).$$

    $x^{\alpha-1}(1-x)^{\alpha-1} \Rightarrow (0.5-\Delta)^{\alpha-1}(0.5+\Delta)^{\alpha-1}$, $(0.5+\Delta)^{\alpha-1}(0.5-\Delta)^{\alpha-1}$

    When $\alpha=\beta$: $E(X)=\frac{\alpha}{\alpha+\beta}=\frac{1}{2}$, $Var(X) = \frac{1}{4(2\alpha+1)}$, $Var(X)\downarrow$ $\alpha\uparrow$

    - As the common value of α and β increases, the distribution becomes more peaked at $x=0.5$ and there is less probability outside of the central portion.



- When $\beta>\alpha$, values close to 0 become more likely than those close to 1; when $\beta<\alpha$, values close to 1 are more likely than those close to 0 (skewed) ($\because$ pdf larger)

  (**Q**: How to connect it with $E(X)$?) $\frac{\alpha}{\alpha+\beta}$

- Summary for $X \sim$ Beta(α, β)
  - Pdf:

    $$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

  - Cdf: $F(x) = B(x;\alpha,\beta)/B(\alpha,\beta)$.
  - Parameters: α, $\beta > 0$.
  - Mean: $E(X) = \alpha/(\alpha + \beta)$.
  - Variance: $Var(X) = (\alpha\beta)/[(\alpha + \beta)^2(\alpha + \beta + 1)]$.

$E\left(\frac{X-\mu}{\sigma}\right)=0$, $Var\left(\frac{X-\mu}{\sigma}\right)=1$

standardization (標準化)

- Normal (Gaussian) Distribution 高斯 常態
  - For $\mu\in\mathbb{R}$ and $\sigma>0$, the function (fixed constants)

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$$

possible values of X

$\boxed{y=\frac{x-\mu}{\sigma}} \Rightarrow x=\sigma y+\mu$

$\frac{dx}{dy}=\sigma \Rightarrow dx=\sigma\,dy$

(*)

is a pdf since (1) $f(x) \ge 0$ for all $x\in\mathbb{R}$, and (2)

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \underset{\text{by (X) in LNp.6-30}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy \equiv \frac{I}{\sqrt{2\pi}},$$

and

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy\right)$$

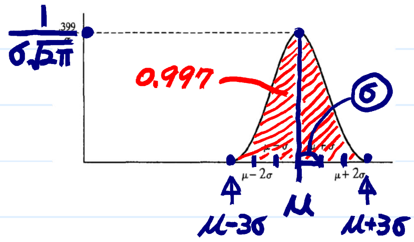
$x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow dxdy = r\,drd\theta$

$\begin{vmatrix} dx/dr & dx/d\theta \\ dy/dr & dy/d\theta \end{vmatrix} = r$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dxdy = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}} r\, d\theta dr$$

$$= 2\pi \int_0^{\infty} r e^{-\frac{r^2}{2}}\, dr = -2\pi e^{-\frac{r^2}{2}}\Big|_0^{\infty} = 2\pi.$$

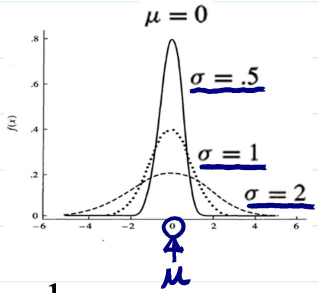
- The distribution of a random variable X with this pdf is called the *normal* (*Gaussian*) distribution with parameters μ and σ, denoted by $N(\mu, \sigma^2)$.



- The normal pdf is a bell-shaped curve.
  - It is symmetric about the point μ, i.e.,

$$f(\mu+\Delta) = f(\mu-\Delta)$$

and falls off in the rate determined by σ.



Solve $\frac{df}{dx} = 0 \Rightarrow x = \mu$ or $\frac{d}{dx}\ln(f) = 0 \Rightarrow x = \mu$

  - The pdf has a maximum at μ (can be shown by differentiation) and the maximum height is $1/(\sigma\sqrt{2\pi})$.