- Notice that it does not mean the two events $\{X > s+t\}$ and $\{X > s\}$ are independent.

If $\{X>s+t\}$ & $\{X>s\}$ are independent, $P(X>s+t|X>s) \neq P(X>s+t)$

➢ Summary for $X \sim$ Exponential(λ)

- Pdf: $f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$
- Cdf: $F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$
- Parameters: $\lambda > 0$.
- Mean: $E(X) = 1/\lambda$.
- Variance: $Var(X) = 1/\lambda^2$.

λ: $\frac{次}{單位時間}$ λ': $\frac{單位時間}{次}$

alternative exponential ($\lambda' = \frac{1}{\lambda}$)
- pdf: $f(x) = \frac{1}{\lambda'} e^{-\frac{x}{\lambda'}}$, if $x \geq 0$
- cdf: $F(x) = 1 - e^{-\frac{x}{\lambda'}}$, if $x \geq 0$
- $E(X) = \lambda'$
- $Var(X) = \lambda'^2$

$\boxed{\lambda = \frac{1}{\lambda'}}$

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- Gamma Distribution ← Recall LNp.6-19

伽瑪

➢ Gamma Function

- Definition. For $\alpha > 0$, the *gamma function* is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

- a recursive formula
- know $\Gamma(\alpha)$ for $\alpha \in (0,1]$
  ⇒ know $\Gamma(\alpha)$ on any α

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$ (exercise)

check textbook, CH5. Theoretical exercise, 5.21

- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

Proof. By integration by parts,

$$\Gamma(\alpha+1) = \int_0^\infty \underbrace{x^\alpha}_{f} \underbrace{e^{-x}\, dx}_{g'}$$

$$= -x^\alpha e^{-x}\Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x}\, dx = \alpha\Gamma(\alpha).$$

- $\Gamma(\alpha) = (\alpha-1)!$ if α is an integer

Proof. $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) = (\alpha-1)(\alpha-2)\Gamma(\alpha-2) = \cdots$

$$= (\alpha-1)(\alpha-2)\cdots\Gamma(1) = (\alpha-1)!$$

- $\Gamma(\alpha/2) = \frac{\sqrt{\pi}(\alpha-1)!}{2^{\alpha-1}(\frac{\alpha-1}{2})!}$ if α is an odd integer

Proof. $\Gamma(\frac{\alpha}{2}) = (\frac{\alpha-2}{2})\Gamma(\frac{\alpha}{2}-1) = \cdots = (\frac{\alpha-2}{2})(\frac{\alpha-4}{2})\cdots\frac{1}{2}\Gamma(\frac{1}{2})$

$\alpha-1$, $\alpha-3$, … ; $\frac{\alpha-1}{2}$, $\frac{\alpha-3}{2}$, … ; 2, 2

- Gamma function is a generalization of the factorial functions

➢ For $\alpha, \lambda > 0$, the function

fixed

from Gamma function

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

possible values of X

cf. waiting time (LNp.6-19)

is a pdf since (1) $f(x) \geq 0$ for all $x \in \mathbb{R}$, and (2)

$$\int_{-\infty}^\infty f(x)\, dx = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx$$

$y = \lambda x \Rightarrow x = \frac{1}{\lambda} y$

$\frac{dx}{dy} = \frac{1}{\lambda} \Rightarrow dx = \frac{1}{\lambda} dy$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y}\, dy = 1.$$

- The distribution of a random variable X with this pdf is called the *gamma* distribution with parameters α and λ. not necessary integers

➢ The cdf of gamma distribution can be expressed in terms of the *incomplete gamma function*, i.e., $F(x)=0$ for $x<0$, and for $x \geq 0$,

$$F(x) = \int_0^x \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} dy$$

$z=\lambda y \Rightarrow y = z/\lambda$, $\frac{dy}{dz} = \frac{1}{\lambda} \Rightarrow dy = \frac{1}{\lambda} dz$

$$\stackrel{\downarrow}{=} \frac{1}{\Gamma(\alpha)} \int_0^{\lambda x} \underbrace{z^{\alpha-1}}_{f} \underbrace{e^{-z}}_{g'} dz \equiv \frac{1}{\Gamma(\alpha)} \cdot \gamma(\alpha, \lambda x)$$

If α is an integer

$F(x) \stackrel{\text{(exercise)}}{=} 1 - \sum_{k=0}^{\alpha-1} \frac{e^{-\lambda x}(\lambda x)^k}{k!}$

$P(X \leq x) = F_X(x) = 1 - F_Y(\alpha - 1)$, with $X \sim$ Gamma(α,λ), $P(Y \geq \alpha)$, $Y \sim$ Poisson(λx)

discrete time version (LNp.6-19)

$P(X \leq n) = P(Y \geq r)$, with $X \sim$ negative binomial (r, p), $Y \sim$ binomial (n, p)

$P(X > n) = P(Y < r)$ (LNp 5-28)

➢ Theorem. The mean and variance of a gamma distribution with parameter α and λ are

Intuitive explanation

$$\mu = \alpha/\lambda \quad \text{and} \quad \sigma^2 = \alpha/\lambda^2.$$

$\mu = \alpha \cdot \left(\frac{1}{\lambda}\right)$, $\sigma^2 = E(X^2) - [E(X)]^2$

$Var(T_1 + \cdots + T_\alpha) = \sum_{i=1}^{\alpha} Var(T_i)$ (indep. $T_1, \ldots, T_\alpha$) (LNp.8-13)

Proof.

$$E(X) = \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \, dx$$

$$= \frac{\cancel{\lambda^\alpha}}{\cancel{\Gamma(\alpha)}} \frac{\Gamma(\alpha+1)}{\lambda^{\cancel{\alpha}+1}} \int_0^\infty \underbrace{\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x}}_{\text{pdf of Gamma}(\alpha+1,\lambda)} \, dx = \frac{\alpha}{\lambda}.$$

$\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$

$$E(X^2) = \int_0^\infty x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \, dx$$

$$= \frac{\cancel{\lambda^\alpha}}{\cancel{\Gamma(\alpha)}} \frac{\Gamma(\alpha+2)}{\lambda^{\cancel{\alpha}+2}} \int_0^\infty \underbrace{\frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)} x^{(\alpha+2)-1} e^{-\lambda x}}_{\text{pdf of Gamma}(\alpha+2,\lambda)} \, dx = \frac{\alpha(\alpha+1)}{\lambda^2}$$

$\Gamma(\alpha+2) = (\alpha+1)\alpha\Gamma(\alpha)$

- (exercise) $E(X^k) = \frac{\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)}$, for $0 < k$, and

X^{-k}

$$E\left(\frac{1}{X^k}\right) = \frac{\lambda^k \Gamma(\alpha - k)}{\Gamma(\alpha)}, \quad \text{for } 0 < k < \alpha.$$

➢ Some properties

check the graph in LNp.6-19

- The gamma distribution can be used to model the waiting time until a number of random events occurs

the number = α (integer)

  - When α=1, it is exponential(λ)

prove in LNp.7-33 or using mgf (chapter 7)

  - $T_1, \ldots, T_n$: n independent exponential(λ) r.v.'s
    $\Rightarrow T_1 + \cdots + T_n \sim$ Gamma(n, λ)
  - Gamma distribution can be thought of as a continuous analogue of the negative binomial distribution

- A summary

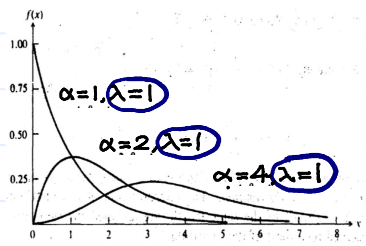
(check LNp 6-19)

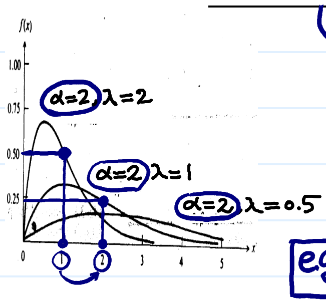
	Discrete Time Version	Continuous Time Version
number of events	binomial	Poisson
waiting time until 1st event occurs (memoryless)	geometric	exponential
waiting time until rth events occur	negative binomial	gamma

- α is called shape parameter and λ scale parameter ← Why?
  (**Q**: how to interpret α and λ in terms of waiting time?)

X ~ Gamma(α, λ)
aX ~ Gamma(α, λ/a)
for a > 0.
(exercise, can use the Thm in LNp.6-10)

α: X is waiting time until αth occurance
λ: 次/單位時間





eg. $X_1 \to \lambda_1$: 次/天
$X_2 \to \lambda_2$: 次/小時
$X_3 \to \lambda_3$: 次/分鐘
$\lambda_1 = 24\lambda_2 = 1440\lambda_3$
$X_1 = \frac{X_2}{24} = \frac{X_3}{1440}$

FYI

- A special case of the gamma distribution occurs when $\alpha = n/2$ and $\lambda = 1/2$ for some positive integer n. This is known as the Chi-squared distribution with n degrees of freedom (Chapter 6)

definition: $X_1^2 + \cdots + X_n^2$ where $X_1, \cdots, X_n$ are independent and $\sim$ Normal(0,1)

➢ Summary for $X \sim$ Gamma(α, λ)
- Pdf: $$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$
- Cdf: $F(x) = \gamma(\alpha, \lambda x)/\Gamma(\alpha)$.
- Parameters: α, λ > 0.
- Mean: $E(X) = \alpha/\lambda$.
- Variance: $Var(X) = \alpha/\lambda^2$.