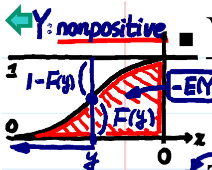


relation btn E(X) & cdf
cf. relation btn E(X) & pmf/pdf

■ Variance of Linear Function. For $a, b \in \mathbb{R}$, fixed constant

$$Var(aX+b) = a^2 \cdot Var(X)$$ gone

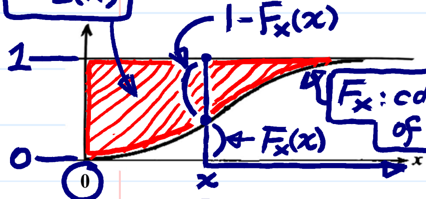
proof. exercise

X=-Y.
X: nonnegative
$1-F_X(x) = F_Y(-x)$

■ Theorem. For a nonnegative continuous random variable X,

$$E(X) = \int_0^\infty 1 - \underbrace{F_X(x)}_{\text{cdf of } X} dx = \int_0^\infty P(X > x) dx.$$

area = E(X)

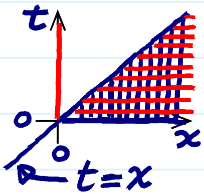


Proof.

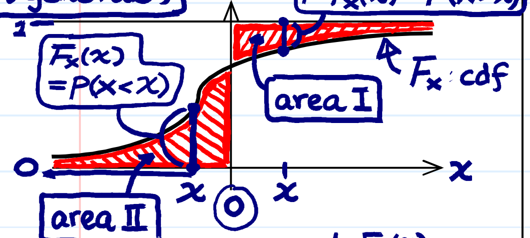
$$\begin{aligned} E(X) &= \int_0^\infty x \cdot f_X(x)\, dx \\ &= \int_0^\infty \left(\int_0^x 1\, dt\right) f_X(x)\, dx \\ &= \int_0^\infty \int_0^x f_X(x)\, dt\, dx \\ &= \int_0^\infty \int_t^\infty f_X(x) dx\, dt = \int_0^\infty 1 - F_X(t) dt. \end{aligned}$$

∵ converge absolutely

$F_X(x)|_t^\infty$



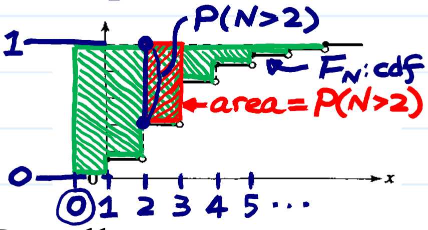
In general,

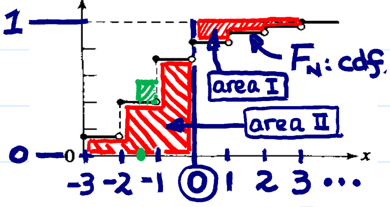


$$E(X) = \int_0^\infty \overbrace{P(X > x)}^{1-F_X(x)} dx - \int_{-\infty}^0 \overbrace{P(X < x)}^{F_X(x)} dx$$

homework

= area I − area II



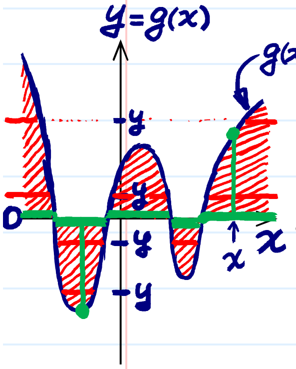


Recall. CH4, Theoretical Exercise #5 (textbook), Let N be a nonnegative integer-valued r.v.,

$$E(N) = \sum_{i=1}^\infty P\{N \geq i\} = \sum_{i=0}^\infty P\{N > i\}$$

■ Proof for the expectation of transformation (LNp.6-13). Let $Y=g(X)$. It holds because

no need to assume g has inverse or g is piecewise strictly monotone



$$\begin{aligned} &\int_0^\infty \underbrace{P(Y > y)}_{>0} dy \\ &= \int_0^\infty P(g(X) > y)\, dy \overset{\text{by the definition of pdf (LNp.6-3)}}{=} \int_0^\infty \left[\int_{\{x: g(x) > y\}} f_X(x)\, dx\right] dy \\ &= \int_{\{x: g(x) > 0\}} \left[\int_0^{g(x)} f_X(x)\, dy\right] dx = \int_{\{x: g(x) > 0\}} g(x) f_X(x)\, dx \end{aligned}$$

∵ converge absolutely

and

$$\begin{aligned} &-\int_{-\infty}^0 \underbrace{P(Y < y)}_{<0} dy \\ &= -\int_{-\infty}^0 P(g(X) < y)\, dy = -\int_{-\infty}^0 \left[\int_{\{x: g(x) < y\}} f_X(x)\, dx\right] dy \\ &= -\int_{\{x: g(x) < 0\}} \left[\int_{g(x)}^0 f_X(x)\, dy\right] dx = +\int_{\{x: g(x) < 0\}} +g(x) f_X(x)\, dx \end{aligned}$$

∵ converge absolutely

$$\int_{-\infty}^{\infty} = \int_{\{g(x)>0\}} + \int_{\{g(x)<0\}}$$

➢ Example (Uniform Distributions)

$$E(\underset{Y}{X^2}) = \int_\alpha^\beta \frac{x^2}{\beta-\alpha} dx = \frac{1}{3} \frac{x^3}{\beta-\alpha}\Big|_\alpha^\beta = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)} = \frac{\beta^2+\alpha\beta+\alpha^2}{3}$$

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 = \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \left(\frac{\alpha+\beta}{2}\right)^2 \\ &= \frac{4(\beta^2+\alpha\beta+\alpha^2) - 3(\beta^2+2\alpha\beta+\alpha^2)}{12} = \frac{(\beta-\alpha)^2}{12}. \end{aligned}$$

若 $\beta - \alpha \uparrow$, $Var(X) \uparrow$

❖ **Reading**: textbook, Sec 5.1, 5.2, 5.3, 5.7

Some Commonly Used Continuous Distributions

a uniform r.v. can only take values in a finite interval (α, β)

original sample space Ω=?

- Uniform Distribution

均匀

important in pseudo-random number generation (LNp.6-8~9)

➤ Summary for $X \sim$ Uniform(α, β)

- Pdf: $$f(x) = \begin{cases} 1/(\beta-\alpha), & \text{if } \alpha < x \le \beta, \\ 0, & \text{otherwise,} \end{cases}$$
- Cdf: $$F(x) = \begin{cases} 0, & \text{if } x \le \alpha, \\ (x-\alpha)/(\beta-\alpha), & \text{if } \alpha < x \le \beta, \\ 1, & \text{if } x > \beta. \end{cases}$$
- Parameters: $-\infty < \alpha < \beta < \infty$
- Mean: $E(X) = (\alpha+\beta)/2$
- Variance: $Var(X) = (\beta-\alpha)^2/12$

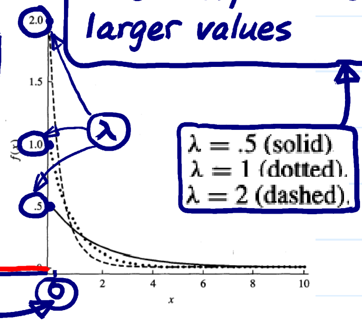
- Exponential Distribution

指數

possible values of X

The pdf is discontinuous at $x=0$ $\Leftrightarrow$ $\frac{dF}{dx}$ does not exist at $x=0$

When $\lambda\downarrow$, X is more likely to have larger values



➤ For $\lambda>0$, the function

fixed

P(X>0)=1

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0, \end{cases}$$

is a pdf since (1) $f(x) \ge 0$ for all $x \in$ R, and (2)

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{\infty} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^{\infty} = 1.$$

- The distribution of a random variable X with this pdf is called the *exponential* distribution with parameter λ.

➤ The cdf of an exponential r.v. is $F(x)=0$ for $x<0$, and for $x \ge 0$,

check $\frac{dF}{dx}=f$ (exercise)

$$F(x) = P(X \le x) = \int_0^x \lambda e^{-\lambda y}\, dy = -e^{-\lambda y}\Big|_0^x = 1 - e^{-\lambda x}.$$

➤ Theorem. The mean and variance of an exponential distribution with parameter λ are

$\lambda\downarrow\ \mu\uparrow$ Check the graph in LNp.6-17 $\lambda\downarrow\ \sigma^2\uparrow$

Intuitive interpretation (see LNp.6-19)

$$\mu = 1/\lambda \quad\text{and}\quad \underset{\substack{\parallel \\ E(X^2)-[E(X)]^2}}{\sigma^2} = 1/\lambda^2.$$

Proof.

$y=\lambda x \Rightarrow x=y/\lambda \Rightarrow \frac{dx}{dy}=\frac{1}{\lambda} \Rightarrow dx=\frac{1}{\lambda}dy$

check LNp.6-23 $\Gamma(n)=(n-1)!$

$$\begin{aligned} E(X) &= \int_0^{\infty} x\lambda e^{-\lambda x} dx = \int_0^{\infty} \frac{y}{\cancel{\lambda}}(\cancel{\lambda} e^{-y})\frac{1}{\lambda}dy \\ &= \frac{1}{\lambda}\underbrace{\int_0^{\infty} y e^{-y} dy}_{\text{LNp.22}} = \frac{1}{\lambda}\Gamma(2) = \frac{1}{\lambda}. \\ E(X^2) &= \int_0^{\infty} x^2\lambda e^{-\lambda x} dx = \int_0^{\infty} \left(\frac{y}{\lambda}\right)^2 (\cancel{\lambda} e^{-y})\frac{1}{\cancel{\lambda}}dy \\ &= \frac{1}{\lambda^2}\underbrace{\int_0^{\infty} y^2 e^{-y} dy}_{\text{LNp.6-22}} = \frac{1}{\lambda^2}\Gamma(3) = \frac{2}{\lambda^2}. \end{aligned}$$

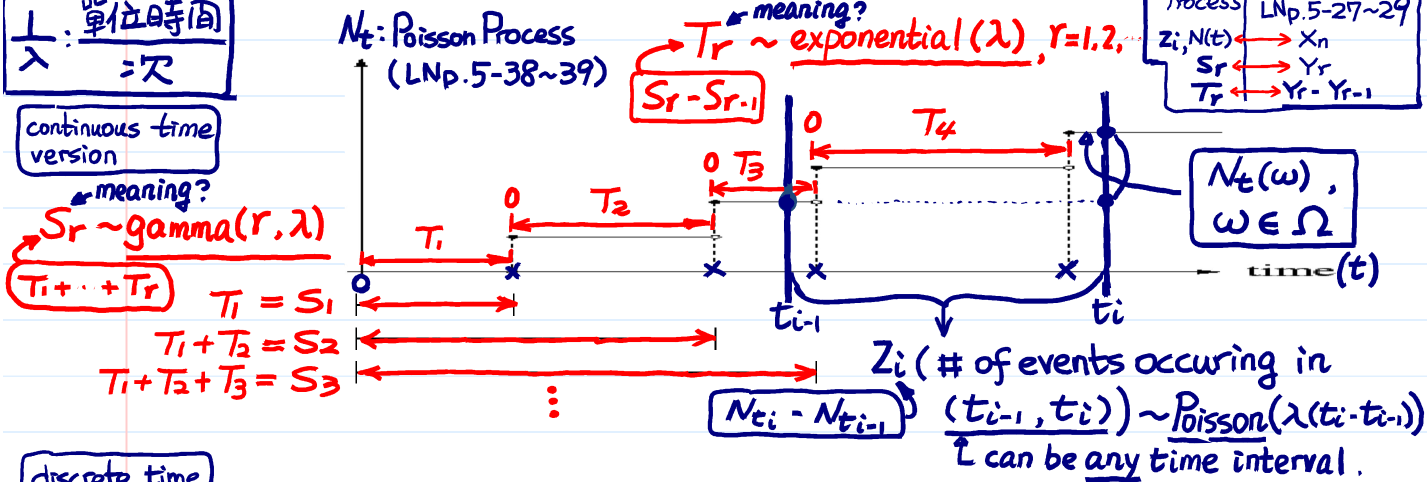
➤ Some properties

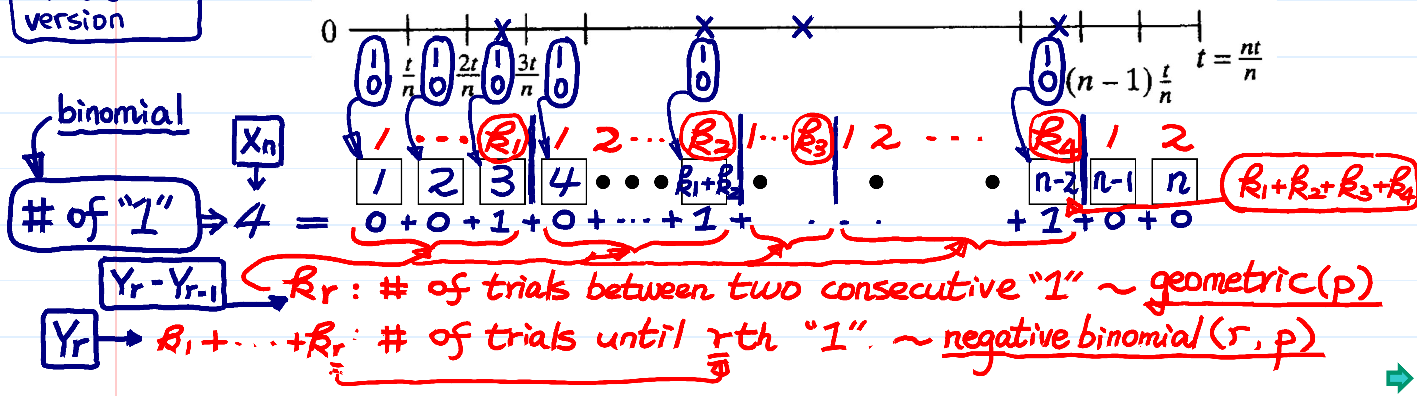
- The exponential distribution is often used to model the length of waiting time until an event occurs or the lifetime

λ: 次/單位時間

$\frac{1}{\lambda}$: 單位時間/次

- The parameter λ is called the *rate* and is the average number of events that occur in unit time. (This gives an intuitive interpretation of $E(X)=1/\lambda$.)



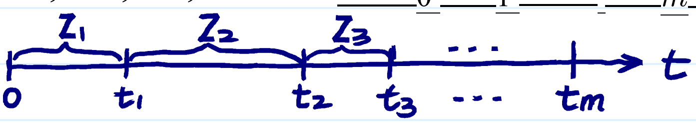


- **Theorem** (relationship between exponential, gamma, and Poisson distributions, Sec. 9.1, textbook). Let

Problem Formulation in LNp.5-27~28 (c.f.)

1. $T_1, T_2, T_3, \ldots,$ be independent and $\sim$ exponential(λ),
2. $S_r = T_1 + \cdots + T_r$, $r=1, 2, 3, \ldots,$
3. Z_i be the number of S_r's that falls in the time interval $(t_{i-1}, t_i]$, $i=1, \ldots, m$, where $0 = t_0 < t_1 < \cdots < t_m < \infty$.

Then,



(i) $S_r \sim$ gamma(r, λ). ← prove in Chapter 7

(ii) $Z_1, \ldots, Z_m$ are independent,

(iii) $Z_i \sim$ Poisson$(\lambda(t_i - t_{i-1}))$,

(iv) The reverse statement is also true.

1. $Z_1, \ldots, Z_m, \ldots$: independent & $Z_i \sim$ Poisson$(\lambda(t_i - t_{i-1}))$
2. define $S_r \to$ define T_r
3. Then, $T_r \overset{\text{indep.}}{\sim}$ exponential(λ)

- The rate parameter λ is the same for the Poisson, exponential, and gamma random variables.

$N_t = Z$ on $[0, t]$ $\sim$ Poisson(λt)

$1 - F_{T_1}(t) = P(T_1 > t) = P(Z = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$

- The exponential distribution can be thought of as the continuous analogue of the geometric distribution.

The reverse is also true.

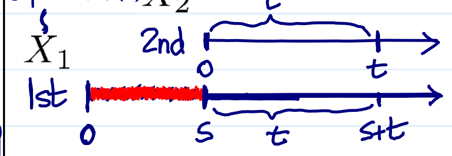
Theorem. The exponential distribution (like the geometric distribution) is *memoryless*, i.e., for $s, t \geq 0$,

e.g. X=lifetime ~exponential

$$P(\overset{X-s>t}{X > s+t} \mid X > s) = P(X > t). \quad 1-F_X(t)$$

where $X \sim$ exponential(λ).

X_1 exponential(λ) X_2 ~ exponential(λ)



$P(X_1 > s+t \mid X_1 > s)$
$= P(X_1 > t) = P(X_2 > t)$

Proof. Compare the difference: ① $X_1 = X_2$ ② X_1 dist. = X_2 dist. ($X_1 \sim X_2$)

$$P(X > s+t \mid X > s) = \frac{P(\{X > s+t\} \cap \{X > s\})}{P(\{X > s\})} = \frac{P(\{X > s+t\})}{P(\{X > s\})}$$

$$= \frac{1 - F_X(s+t)}{1 - F_X(s)} = \frac{e^{-\lambda(\cancel{s}+t)}}{e^{-\lambda \cancel{s}}} = e^{-\lambda t} = P(X > t)$$

by the cdf given in LNp 6-18

Intuitive explanation: check the graph in LNp.6-19.

- This means that the distribution of the waiting time to the next event remains the same regardless of how long we have already been waiting.
- This only happens when events occur (or not) totally at random, i.e., independent of past history. ← connection between exponential & geometric

Q: Why negative binomial (gamma) with $r > 1$ does not possess the memoryless property?

- Notice that it does not mean the two events $\{X > s+t\}$ and $\{X > s\}$ are independent.

If $\{X>s+t\}$ & $\{X>s\}$ are independent, $P(X>s+t \mid X>s) \neq P(X>s+t)$

➢ Summary for $X \sim$ Exponential(λ)

- Pdf: $f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$
- Cdf: $F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$
- Parameters: $\lambda > 0$.
- Mean: $E(X) = 1/\lambda$.
- Variance: $Var(X) = 1/\lambda^2$.

$\lambda: \frac{次}{單位時間}$ $\lambda': \frac{單位時間}{次}$

alternative exponential ($\lambda' = \frac{1}{\lambda}$)
- pdf: $f(x) = \frac{1}{\lambda'} e^{-\frac{x}{\lambda'}}$, if $x \geq 0$
- cdf: $F(x) = 1 - e^{-\frac{x}{\lambda'}}$, if $x \geq 0$
- $E(X) = \lambda'$
- $Var(X) = \lambda'^2$

$\boxed{\lambda = \frac{1}{\lambda'}}$

11/14

- **Gamma Distribution** ← Recall LNp.6-19

伽瑪

➢ Gamma Function

- Definition. For $\alpha > 0$, the *gamma function* is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx.$$

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$ (exercise)
- $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

check textbook, CH5. Theoretical exercise. 5.21