- The cdf of normal distribution does not have a close form. (usually, denoted by Φ)
- Theorem. The mean and variance of a $N(\mu, \sigma^2)$ distribution are μ and σ^2, respectively. (Intuition; $=E(X^2)-[E(X)]^2$; Why? Check the graphs in LNp.6-31 & ★ in LNp.6-30)
  - μ: location parameter; σ (or σ^2): scale (or dispersion) parameter

Proof.

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \overset{\text{by (*) in LNp.6-30}}{=} \int_{-\infty}^{\infty} (\sigma y + \mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}}\, dy + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy \quad \left(-e^{-y^2/2}\Big|_{-\infty}^{\infty}\right)$$

$$= \frac{\sigma}{\sqrt{2\pi}} \cdot 0 + \mu \cdot 1 = \mu. \quad (\text{pdf of } N(0,1))$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \overset{\text{by (*) in LNp.6-30}}{=} \int_{-\infty}^{\infty} (\sigma y + \mu)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

$$= \sigma^2 \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy + \frac{2\mu\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}}\, dy + \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy \quad (\text{pdf of } N(0,1))$$

$$= \sigma^2 \cdot 1 + \frac{2\mu\sigma}{\sqrt{2\pi}} \cdot 0 + \mu^2 \cdot 1 = \sigma^2 + \mu^2.$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y \left(y e^{-\frac{y^2}{2}}\right) dy = \frac{1}{\sqrt{2\pi}} y \left(-e^{-y^2/2}\right)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left(+e^{-\frac{y^2}{2}}\right) dy$$

(f, g'; pdf of $N(0,1)$)

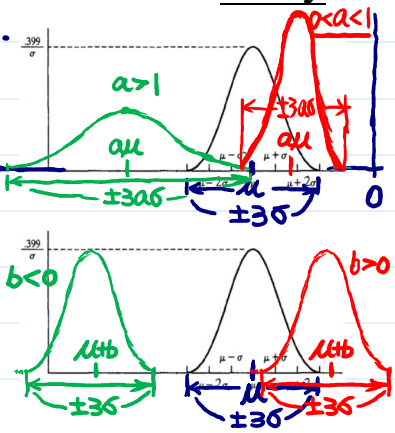
- Some properties (Why? one reason is the central limit thm (CLT))
  - Normal distribution is one of the most widely used distribution. It can be used to model the distribution of many natural phenomena. (∵ bell-shaped distribution; e.g. height, weight, error, ...)
  - Theorem. Suppose that $X \sim N(\mu, \sigma^2)$. The random variable

$$Y = aX + b,$$

where $a \neq 0$, is also normally distributed with parameters $a\mu+b$ and $a^2\sigma^2$, i.e.,

$$Y \sim N(a\mu+b, a^2\sigma^2).$$

(Recall: $E(Y)=aE(X)+b$, $Var(Y)=a^2Var(X)$; Note: $E(Z)=0$, $Var(Z)=1$ for any r.v. X. But, X & Z may not belong to same distribution in general.)



(c.f. graphs in LNp.5-16, 5-18)

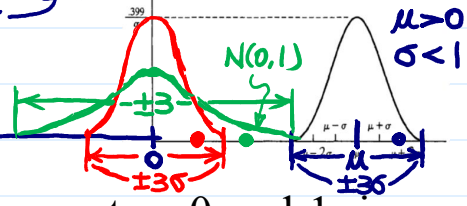
Proof. (by the example in LNp.6-11)

$$f_Y(y) \overset{\sigma_Y}{=} f_X\left(\frac{y-b}{a}\right) \frac{1}{|a|} = \frac{1}{\sqrt{2\pi}|a|\sigma} e^{-\frac{[y-(a\mu+b)]^2}{2\sigma^2 a^2}}. \quad (\mu_Y,\ \sigma_Y^2)$$

  - Corollary. If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$

(standardization (check ★ in LNp.6-30) → remove unit; meaning of $Z=1, 1.5, 2$?)



is a normal random variable with parameters 0 and 1, i.e., $N(0, 1)$, which is called *standard normal distribution*.

- The $N(0, 1)$ distribution is very important since properties of any other normal distributions can be found from those of the standard normal.
  - *The cdf of $N(0, 1)$ is usually denoted by Φ.* $\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$ — no close form
  - Theorem. Suppose that $X\sim N(\mu, \sigma^2)$. The cdf of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

$P(X\le x)=P\left(\frac{X-\mu}{\sigma}\le\frac{x-\mu}{\sigma}\right)$, $\frac{X-\mu}{\sigma}=Z$

Proof. $F_X(x) = F_Z\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$. ($Z\sim N(0,1)$)

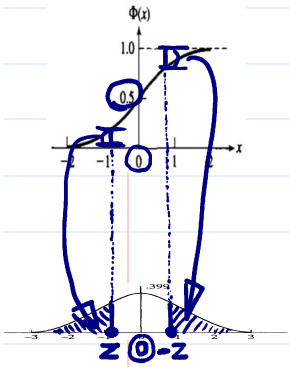
  - Example. Suppose that $X\sim N(\mu, \sigma^2)$. For $-\infty<a<b<\infty$,

$$\begin{aligned} P(a < X < b) &= P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) \\ &= P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) \quad (Z\sim N(0,1)) \\ &= P\left(Z < \frac{b-\mu}{\sigma}\right) - P\left(Z < \frac{a-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \end{aligned}$$

  - Table 5.1 in textbook gives values of Φ.
    To read the table:

$\Phi(x)$: cdf of $N(0,1)$

1. Find the first value of x up to the first place of decimal in the left hand column.
2. Find the second place of decimal across the top row.
3. The value of $\Phi(x)$ is where the row from the first step and the column from the second step intersect.



$\phi(x)$: pdf of $N(0,1)$

$\Phi(z)+\Phi(-z)=1$

$\Phi(0)=\frac{1}{2}$

$\Phi(0.22)=0.5871$

$\Phi(3.36)=0.9996$

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
⋮										
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

- For the values greater than z=3.49, $\Phi(z) \approx 1$.
- For negative values of z, use $\Phi(z)=1-\Phi(-z)$

gamma ↔ exponential, geometric, negative binomial ↔ Bernoulli, binomial

$Z=X_1+X_2+\cdots+X_n$, n is large ($\overset{d}{\approx}$)

- Normal distribution plays a central role in the limit theorems of probability (e.g., Central Limit Theorem, CLT, chapter 8)

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Normal approximation to the Binomial

an example of CLT

(LNp.5-43~44, $N_i\uparrow\infty, R_i\uparrow\infty, \frac{R_i}{N_i}\to p$)

(LNp.5-31~32, $n\uparrow\infty, p_n\downarrow 0, \lambda=np_n$)

- Recall. Poisson & Hypergeometric approximations to Binomial
- Theorem. Suppose that X_n~binomial(n, p).

standardization

sum of n independent Bernoulli(p)

Define $Z_n \triangleq (X_n - np)/\sqrt{np(1-p)}$.

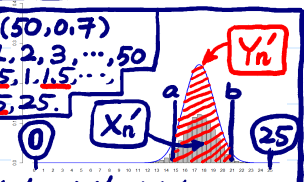
μ_n σ_n^2

p is fixed cf.

$\mu_n=E(X_n)=np$
$\sigma_n^2=Var(X_n)=np(1-p)$

$E(Z_n)=0$
$Var(Z_n)=1$

$X_n \sim B(50,0.7)$
$X_n=0,1,2,3,\cdots,50$
$X_n'=0,0.5,1,1.5,\cdots,24.5,25.$



$Q: X_n'=X_n/2,\ Y_n'=Y_n/2.$
pmf ≈ pdf? (No)
cdf ≈ cdf? (Yes)
$P(a<X_n'\le b)\approx P(a<Y_n'\le b)$? (Yes)

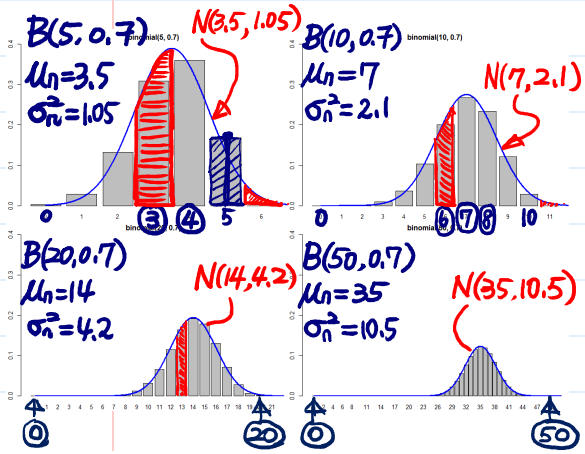
Then, as $n\to\infty$, the distribution of Z_n converge to the $N(0, 1)$ distribution, i.e.,

cdf of Z_n — cdf of $N(0,1)$

$$F_{Z_n}(z) = P(Z_n \le z) \to \Phi(z).$$

$X_n=\sigma_n Z_n+\mu_n$

Proof. It is a special case of the CLT in Chapter 8.



- Plot the pmf of X_n~binomial(n, p)
- Superimpose the pdf of $Y_n \sim N(\mu_n, \sigma_n^2)$ with $\mu_n=np$ and $\sigma_n^2=np(1-p)$.
- When n is sufficiently large, the normal pdf approximates the binomial pmf.
- $Z_n \overset{d}{\approx} (Y_n-\mu_n)/\sigma_n$ ($Y_n \sim N(0,1)$)

cf. $Z_n \approx (Y_n-\mu_n)/\sigma_n$

not a rigorous statement

- The size of n to achieve a good approximation depends on the value of p.
  - For p near 0.5 ⇒ moderate n is enough
  - For p close to zero or one ⇒ require much larger n (skewed)

Why? Check LNp.5-22 graphs.

Q: Poisson(λ) $\overset{d}{\approx}$ Normal?
Ans. Yes, when λ is large. ($\lambda \leftrightarrow np$)

- Continuity Correction (for integer-valued discrete r.v.'s)
  - Q: Why need continuity correction?

    Ans. The binomial(n, p) is a discrete r.v. and we are approximating it with a continuous r.v.. (cf.)

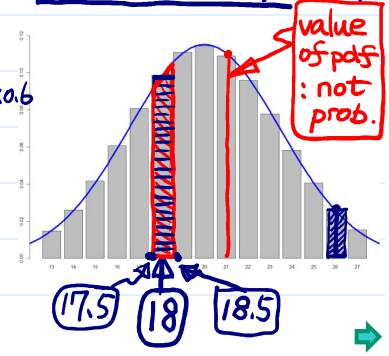
    distribution — Normal

    (height)
    pmf: value→prob.
    pdf: area→prob.

    - For example, suppose X~binomial$(50, 0.4)$ and we want to find $P(X=18)$, which is larger than 0.
    - With the normal r.v. $Y \sim N(20, 12)$, however, $P(Y=18)=0$ because Y has a continuous distribution

    50×0.4 — $50\times0.4\times0.6$

    How about using the value of pdf $f_Y(18)$?



  - Instead, we make a continuity correction,

$$\begin{aligned} P(X=18) &= P(17.5<X<18.5) \\ &= P\left(\frac{17.5-(50\cdot0.4)}{\sqrt{50\cdot0.4\cdot0.6}} < Z_n < \frac{18.5-(50\cdot0.4)}{\sqrt{50\cdot0.4\cdot0.6}}\right) \\ &\approx P\left(\frac{17.5-(50\cdot0.4)}{\sqrt{50\cdot0.4\cdot0.6}} < Z < \frac{18.5-(50\cdot0.4)}{\sqrt{50\cdot0.4\cdot0.6}}\right) \\ &= P\left(-\frac{2.5}{\sqrt{12}} < Z < -\frac{1.5}{\sqrt{12}}\right) = P\left(Z < -\frac{1.5}{\sqrt{12}}\right) - P\left(Z < -\frac{2.5}{\sqrt{12}}\right) \\ &= \Phi\left(-\frac{1.5}{\sqrt{12}}\right) - \Phi\left(-\frac{2.5}{\sqrt{12}}\right) = \left(1-\Phi\left(\frac{1.5}{\sqrt{12}}\right)\right) - \left(1-\Phi\left(\frac{2.5}{\sqrt{12}}\right)\right) \\ &= \Phi\left(2.5/\sqrt{12}\right) - \Phi\left(1.5/\sqrt{12}\right) \end{aligned}$$

by CLT; $Z \sim N(0,1)$

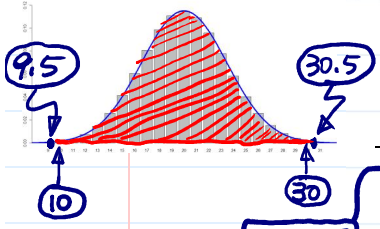
and can obtain the approximate value from Table 5.1.

- Similary,

$$P(X \ge 30) = P(X > 29.5) = P\left(Z_n > \frac{29.5-(50\cdot0.4)}{\sqrt{12}}\right) \approx P(Z > 9.5/\sqrt{12}) = 1-\Phi(9.5/\sqrt{12}).$$

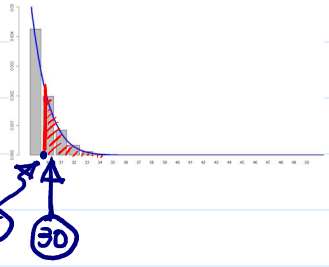
by CLT; $Z \sim N(0,1)$

and



$$\begin{aligned} P(10 \le X \le 30) &= P(9.5 < X < 30.5) \\ &= P\left(\frac{9.5-(50\cdot0.4)}{\sqrt{12}} < Z_n < \frac{30.5-(50\cdot0.4)}{\sqrt{12}}\right) \\ &\approx P\left(-10.5/\sqrt{12} < Z < 10.5/\sqrt{12}\right) \\ &= \Phi\left(10.5/\sqrt{12}\right) - \Phi\left(-10.5/\sqrt{12}\right) \\ &= 2\cdot\Phi\left(10.5/\sqrt{12}\right) - 1 \end{aligned}$$

by CLT; $Z \sim N(0,1)$; $\Phi(-10.5/\sqrt{12}) = 1-\Phi\left(\frac{10.5}{\sqrt{12}}\right)$



➢ Summary for $X \sim$ Normal(μ, σ^2)

- Pdf: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$
- Cdf: no close form, but usually denoted by $\Phi((x-\mu)/\sigma)$.
- Parameters: $\mu \in \mathbb{R}$ and $\sigma > 0$.
- Mean: $E(X) = \mu$.
- Variance: $Var(X) = \sigma^2$.

$y = \left(\frac{x-\nu}{\alpha}\right)^{\beta} \Rightarrow x = \alpha y^{\frac{1}{\beta}} + \nu$

$\frac{dx}{dy} = \frac{\alpha}{\beta} y^{\frac{1}{\beta}-1} \Rightarrow dx = \frac{\alpha}{\beta} y^{\frac{1}{\beta}-1} dy$ (Δ)

• Weibull Distribution

➢ For $\alpha, \beta > 0$ and $\nu \in \mathbb{R}$, the function

fixed constants; possible values of X

$$f(x) = \begin{cases} \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}}, & \text{if } x \ge \nu, \\ 0, & \text{if } x < \nu, \end{cases}$$

is a pdf since (1) $f(x) \ge 0$ for all $x \in \mathbb{R}$, and (2)

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{\nu}^{\infty} \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}} dx \overset{(\Delta)}{=} \int_0^{\infty} e^{-y}\,dy = -e^{-y}\Big|_0^{\infty} = 1.$$

pdf of exponential(1)

- The distribution of a random variable X with this pdf is called the *Weibull* distribution with parameters α, β, and ν.

➢(exercise) The cdf of Weibull distribution is

by (Δ) in LNp.6-39

$$F(x)=\begin{cases}1-e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}}, & \text{if } x\geq\nu,\\ 0, & \text{if } x<\nu.\end{cases}$$

➢Theorem. The mean and variance of a Weibull distribution with parameters α, β, and ν are

$$\mu=\alpha\Gamma\left(1+\frac{1}{\beta}\right)+\nu \quad \text{and}$$

$$\sigma^2=\alpha^2\left\{\Gamma\left(1+\frac{2}{\beta}\right)-\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}.$$

$E(X^2)-[E(X)]^2=$

Proof. $E(X)=\int_v^\infty x\cdot\frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1}e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}}\,dx$

$$\overset{y=\left(\frac{x-\nu}{\alpha}\right)^{\beta}}{=}\int_0^\infty(\alpha y^{1/\beta}+\nu)e^{-y}\,dy$$

$$=\alpha\int_0^\infty y^{(1/\beta+1)-1}e^{-y}\,dy+\nu\int_0^\infty e^{-y}\,dy=\alpha\Gamma\left(\frac{1}{\beta}+1\right)+\nu$$

pdf of exponential(1)

$$E(X^2)=\int_v^\infty x^2\cdot\frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1}e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}}\,dx$$

$$\overset{y=\left(\frac{x-\nu}{\alpha}\right)^{\beta}}{=}\int_0^\infty(\alpha y^{1/\beta}+\nu)^2e^{-y}\,dy$$

$$=\alpha^2\int_0^\infty y^{(2/\beta+1)-1}e^{-y}\,dy+2\alpha\nu\int_0^\infty y^{(1/\beta+1)-1}e^{-y}\,dy+\nu^2\int_0^\infty e^{-y}\,dy$$

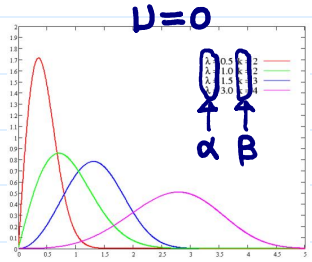
$$=\alpha^2\Gamma\left(\frac{2}{\beta}+1\right)+2\alpha\nu\Gamma\left(\frac{1}{\beta}+1\right)+\nu^2$$

➢Some properties

- Weibull distribution is widely used to model lifetime (cf., exponential)

memoryless

- α: scale parameter; β: shape parameter; ν: location parameter



- Theorem. If X~exponential(λ), then

$$Y=\alpha(\lambda X)^{1/\beta}+\nu$$

Hint: X~Gamma(α,λ) ⇒ aX~Gamma(α, λ/a), a>0 (LNp.6-26)

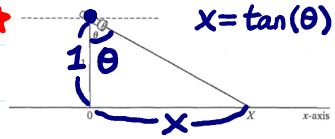
Note: λX~exponential(1)

Thm (LNp.6-10)

is distributed as Weibull with parameters α, β, and ν (exercise).

柯西

• Cauchy Distribution



➢For μ∈ℝ and σ>0, the function

fixed constants

possible values of X

$$f(x)=\frac{\sigma}{\pi}\cdot\frac{1}{\sigma^2+(x-\mu)^2},\quad -\infty<x<\infty,$$

θ~Uniform(−π/2, π/2)
Let X=tan(θ), then
X~Cauchy(0,1)
(exercise)

(∇)

$y=\frac{x-\mu}{\sigma}$
$\Rightarrow x=\sigma y+\mu$
$\frac{dx}{dy}=\sigma$
$\Rightarrow dx=\sigma dy$

is a pdf since (1) $f(x)\geq 0$ for all $x\in\mathbb{R}$, and (2)

$$\int_{-\infty}^\infty f(x)\,dx=\int_{-\infty}^\infty\frac{\sigma}{\pi}\frac{1}{\sigma^2+(x-\mu)^2}\,dx$$

pdf of Cauchy(0,1)

$$\overset{y=\frac{x-\mu}{\sigma}}{=}\int_{-\infty}^\infty\frac{1}{\pi}\frac{1}{1+y^2}\,dy=\frac{1}{\pi}\tan^{-1}(y)\Big|_{-\infty}^{\infty}=1.$$

- The distribution of a random variable X with this pdf is called the *Cauchy* distribution with parameters μ and σ, denoted by Cauchy(μ, σ).

not mean — not standard deviation

Cauchy(μ,σ) ⟵ $Z=F(X)$ ⟵ Uniform(0,1) ⟵ $Z-\frac{1}{2}$ ⟵ $\frac{1}{\pi}$ ⟵ ~Uniform $(-\frac{\pi}{2},\frac{\pi}{2})$

Recall. The 2 Thms in LNp.6-8 & 6-9, and ★ in LNp.6-41.

- The cdf of Cauchy distribution is

$$F(x)=\int_{-\infty}^{x}\frac{\sigma}{\pi}\frac{1}{\sigma^2+(y-\mu)^2}\,dy=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x-\mu}{\sigma}\right)$$

for $-\infty<x<\infty$. (exercise) ← by (▽) in LNp.6-41.

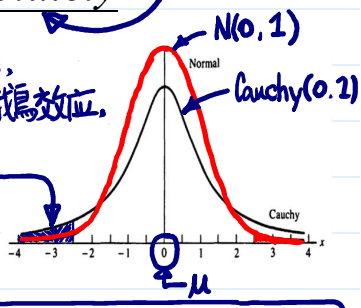
∵ $\int_{-\infty}^{\infty}|x|f(x)dx=\infty$, $\int_{-\infty}^{\infty}x^2f(x)dx=\infty$

- The mean and variance of Cauchy distribution do *not exist* because the integral does *not converge absolutely*

- Some properties

Why? check the graph ★ in LNp.6-41

- Cauchy is a heavy tail distribution → model traffic, wealth, 黑天鵝效應, ...



- μ: location parameter; σ: scale parameter

c.f. ⟷ Normal distribution

- Theorem. If X~Cauchy(μ, σ), then $aX+b$~Cauchy($a\mu+b$, $|a|\sigma$).

Proof. (exercise) ← Thm (LNp.6-10)

Note: a pdf $f(x)\downarrow 0$, when $x\to\infty$ or $x\to-\infty$.
But, $f(x)\downarrow 0$ how fast?

❖ **Reading**: textbook, Sec 5.4, 5.5, 5.6

