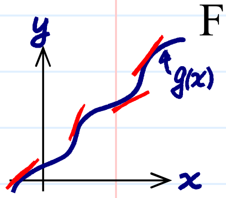
➢ The pdf of Y, denoted by f_Y

Y: continuous r.v. $= g(X)$

1. Suppose that g is a *differentiable* strictly increasing function.

continuous

g^{-1} exists and is also strictly increasing



For $y \in R_Y$, (LNp.6-8)

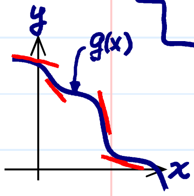
$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(g^{-1}(y))$$

$$= f_X(g^{-1}(y))\underbrace{\frac{dg^{-1}(y)}{dy}}_{>0} = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|.$$

What if $y \notin R_Y$? $\Rightarrow f_Y(y)=0$

2. Suppose that g is a *differentiable* strictly decreasing function.

g^{-1} exists and is also strictly decreasing



For $y \in R_Y$, (LNp.6-8)

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(1 - F_X(g^{-1}(y)))$$

$$= -f_X(g^{-1}(y))\underbrace{\frac{dg^{-1}(y)}{dy}}_{<0} = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|.$$

c.f. Theorem in LNp.5-12

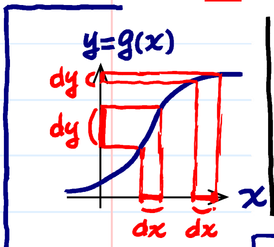
■ **Theorem.** Let X be a continuous random variable with pdf f_X. Let $Y=g(X)$, where g is differentiable and strictly monotone. Then, the pdf of Y, denoted by f_Y, is

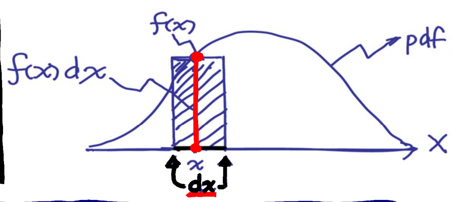
Y: continuous r.v. ?

$$f_Y(y) = f_X(\overbrace{g^{-1}(y)}^{x})\left|\frac{dg^{-1}(y)}{dy}\right|,$$

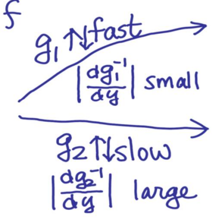
for y such that $y=g(x)$ for some x, and $f_Y(y)=0$ otherwise.

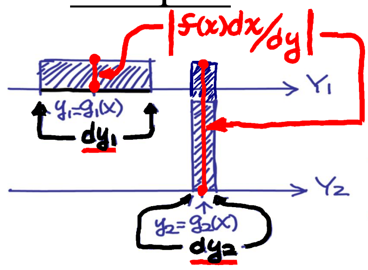
▪ **Q**: What is the role of $|dg^{-1}(y)/dy|$? How to interpret it?





$$\frac{dy}{dx} = g'(x) = \frac{1}{g^{-1\prime}(y)} = \frac{1}{dx/dy}$$





➢ Some Examples. Given the pdf f_X of random variable X,

▪ Find the pdf f_Y of $Y=aX+b$, where $a\neq 0$. (strictly monotone)

$$y = g(x) = ax+b \;\Rightarrow\; x = g^{-1}(y) = \frac{y-b}{a} \;\Rightarrow\; \left|\frac{d}{dy}g^{-1}(y)\right| = \frac{1}{|a|}$$

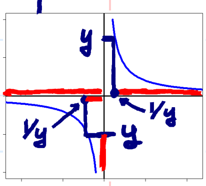
$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{|a|}$$

piecewise strictly monotone & one-to-one

▪ Find the pdf f_Y of $Y=1/X$.

$$y = g(x) = \frac{1}{x} \;\Rightarrow\; x = g^{-1}(y) = \frac{1}{y} \;\Rightarrow\; \left|\frac{d}{dy}g^{-1}(y)\right| = \left|-y^{-2}\right| = \frac{1}{y^2}$$

$$f_Y(y) = f_X\left(\frac{1}{y}\right)\cdot\frac{1}{y^2}$$



For $y\le 0$, $F_Y(y) = P(Y\le y)$
$= P(1/y \le X \le 0) = F_X(0) - F_X(\frac{1}{y})$

For $y>0$, $F_Y(y) = P(Y\le y)$
$= P(\{X\le 0\}\cup\{X\ge 1/y\})$
$= F_X(0) + 1 - F_X(1/y)$

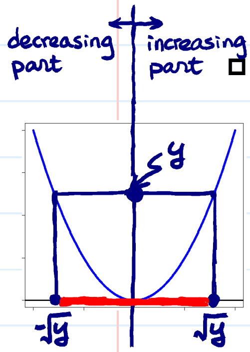
cf. Example in LNp.5-12

◉ Find the cdf F_Y and pdf f_Y of $Y=X^2$. ← piecewise strictly monotone but not one-to-one

□ $$F_Y(y) = P(Y \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= P(X \in (-\infty, \sqrt{y}]) - P(X \in (-\infty, -\sqrt{y}))$$
$$= \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}), & \text{if } y > 0, \\ 0, & \text{if } y \le 0. \end{cases}$$

$[-\sqrt{y}, \sqrt{y}] = (-\infty, \sqrt{y}] \setminus (-\infty, -\sqrt{y})$

increasing part / decreasing part

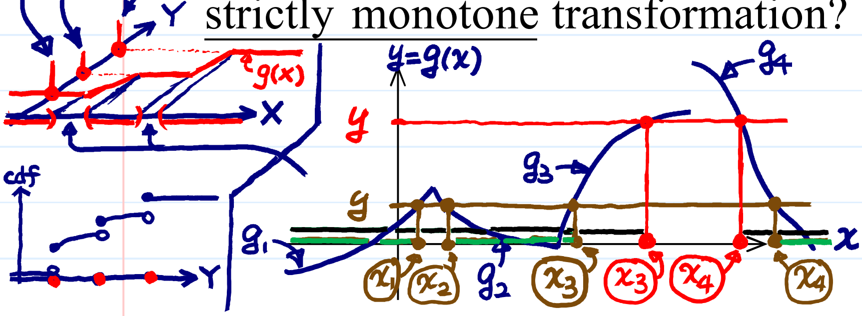


□ For $y > 0$,
$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}[F_X(\sqrt{y}) - F_X(-\sqrt{y})]$$
$$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \frac{-1}{2\sqrt{y}} = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$$

For $y \le 0$, $f_Y(y) = 0$.

▪ **Q**: How to find the cdf F_Y and pdf f_Y for general piecewise strictly monotone transformation? ← finite pieces $g_1, \cdots, g_n$

prob.>0



$$f_Y(y) = \sum_{i=1}^{n} \delta_i \cdot f_X(g_i^{-1}(y)) \left| \frac{d\, g_i^{-1}(y)}{dy} \right|$$

$$\delta_i = \begin{cases} 1, & \text{if } \exists\, x_i \text{ s.t. } g_i(x_i) = y, \\ 0, & \text{otherwise} \end{cases}$$

Q: How to obtain $F_Y(y)$ from $F_X(x)$?

• Expectation, Mean, and Variance

cf. Definition in LNp.5-13

➢ **Definition.** If X has a pdf f_X, then the *expectation* of X is defined by
$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx,$$
provided that the integral converges absolutely.

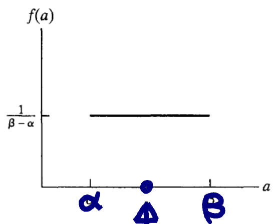
cf. Σ in discrete case

prob. that X near x

$\int_{-\infty}^{\infty} |x| f_X(x) dx < \infty$

cf. equally likely example in LNp.5-13

◉ **Example** (Uniform Distributions). If
$$f_X(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } \alpha < x \le \beta, \\ 0, & \text{otherwise,} \end{cases}$$
then
$$E(X) = \int_\alpha^\beta x \cdot \frac{1}{\beta-\alpha} dx = \frac{1}{2} \cdot \frac{x^2}{\beta-\alpha}\Big|_\alpha^\beta$$
$$= \frac{1}{2} \cdot \frac{\beta^2 - \alpha^2}{\beta - \alpha} = \frac{\alpha+\beta}{2}.$$



➢ Some properties of expectation

no need to assume g has inverse or g is piecewise strictly monotone

cf. discrete case LNp.5-15

◉ **Expectation of Transformation.** If $Y=g(X)$, then
$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx, \equiv E[g(X)]$$
provided that the integral converges absolutely.

by definition

cf. In Calculus, substitution rule: $y = g(x)$ ⇒ $x = g^{-1}(y)$ ⇒ $dx/dy = d/dy\, g^{-1}(y)$

Proof. The proof is given in LNp.6-16

discrete case. LNp.5-16

cf.

- **Expectation of Linear Function.** For $a, b \in \mathbb{R}$, fixed constants

transformation $Y = aX+b$

$$E(aX+b) = a \cdot E(X) + b,$$

since $\underbrace{E(aX+b)}_{E(Y)} = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$

$$= a \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx + b \int_{-\infty}^{\infty} f_X(x)\, dx = a \cdot E(X) + b.$$

c.f. Definition in LNp.5-16

➢ **Definition.** If X has a pdf f_X, then the expectation of X is also called the *mean* of X or f_X and denoted by μ_X, so that fixed value

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx.$$

Σ (discrete case)

The *variance* of X (or f_X) is defined as

deviation of X from μ_X

$$Var(X) = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 \cdot f_X(x)\, dx,$$

fixed constant.

and denoted by σ_X^2. The σ_X is called the *standard deviation*.

fixed value

check LNp.5-14~15 LNp.5-17

➢ Some properties of mean and variance

- The mean and variance for continuous random variables have the *same* intuitive interpretation as in the discrete case.

discrete case LNp.5-19

c.f.

- $Var(X) = E(X^2) - [E(X)]^2$

$\int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx$
$= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2) f_X(x) dx$
$= \int_{-\infty}^{\infty} x^2 f_X(x) dx$
$- 2\mu \int_{-\infty}^{\infty} x f_X(x) dx$
$+ \mu^2 \int_{-\infty}^{\infty} f_X(x) dx$
$= E(X^2) - 2\mu^2 + \mu^2$