

➤ The cdf of normal distribution does not have a close form.

➤ Theorem. The mean and variance of a $N(\mu, \sigma^2)$ distribution are μ and σ^2 , respectively.

Intuition

$$= E(X^2) - [E(X)]^2$$

Why? Check the graphs in LNp.6-31 & ★ in LNp.6-30

■ μ : location parameter; σ (or σ^2): scale (or dispersion) parameter

Proof. $E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \stackrel{\text{by (*) in LNp.6-30}}{=} \int_{-\infty}^{\infty} (\sigma y + \mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$

$$\begin{aligned} &= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}} dy + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= \frac{\sigma}{\sqrt{2\pi}} \cdot 0 + \mu \cdot 1 = \mu. \end{aligned}$$

(Note: $-e^{-y^2/2}$ is the antiderivative of $y e^{-y^2/2}$. The integral of the pdf of $N(0,1)$ is 1.)

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\infty} (\sigma y + \mu)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= \sigma^2 \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy + \frac{2\mu\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{y^2}{2}} dy \\ &\quad + \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \quad \text{pdf of } N(0,1) \\ &= \sigma^2 \cdot 1 + \frac{2\mu\sigma}{\sqrt{2\pi}} \cdot 0 + \mu^2 \cdot 1 = \sigma^2 + \mu^2. \end{aligned}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y \left(y e^{-\frac{y^2}{2}} \right) dy = \frac{1}{\sqrt{2\pi}} y \left(-e^{-y^2/2} \right) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left(+e^{-\frac{y^2}{2}} \right) dy$$

(Note: The first term is 0. The second term is the integral of the pdf of $N(0,1)$, which is 1.)

➤ Some properties Why? one reason is the central limit thm (CLT)

bell-shaped distribution

■ Normal distribution is one of the most widely used distribution. It can be used to model the distribution of many natural phenomena. → e.g. height, weight, error, ...

■ Theorem. Suppose that $X \sim N(\mu, \sigma^2)$. The random variable

Recall.

$$\begin{aligned} E(Y) &= aE(X) + b \\ \text{Var}(Y) &= a^2 \text{Var}(X) \end{aligned}$$

$$Y = aX + b,$$

c.f. graphs in LNp.5-16 5-18

where $a \neq 0$, is also normally distributed with parameters $a\mu + b$ and $a^2\sigma^2$, i.e.,

$$Y \sim N(a\mu + b, a^2\sigma^2).$$

by the example in LNp.6-11

Proof. $f_Y(y) = f_X\left(\frac{y-b}{a}\right) \frac{1}{|a|} = \frac{1}{\sqrt{2\pi}|a|\sigma} e^{-\frac{[y-(a\mu+b)]^2}{2\sigma^2 a^2}}.$

□ Corollary. If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X - \mu}{\sigma} = \frac{1}{\sigma} X - \frac{\mu}{\sigma}$$

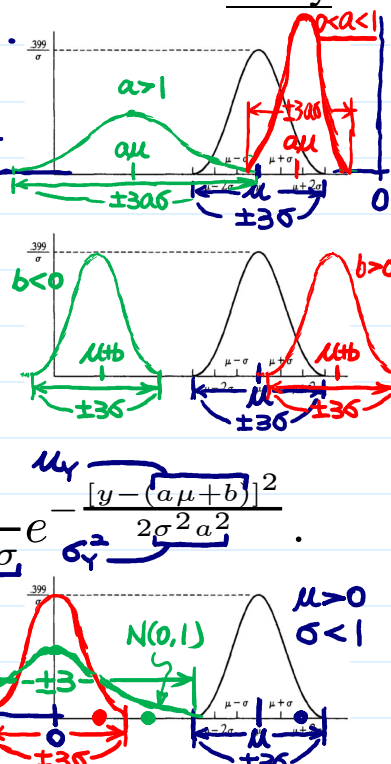
meaning of $Z = 1, 1.5, 2, \dots$

is a normal random variable with parameters 0 and 1, i.e., $N(0, 1)$, which is called standard normal distribution.

Note:

$E(Z) = 0$
 $\text{Var}(Z) = 1$
for any r.v. X .
But, X & Z may not belong to same distribution in general.

standardization (check ★ in LNp.6-30)
→ remove unit



- The $N(0, 1)$ distribution is very important since properties of any other normal distributions can be found from those of the standard normal.

□ The cdf of $N(0, 1)$ is usually denoted by Φ . $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ (no close form)

□ Theorem. Suppose that $X \sim N(\mu, \sigma^2)$. The cdf of X is

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) \quad F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Proof. $F_X(x) = F_Z\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$.
 $\downarrow$
 $N(0,1)$

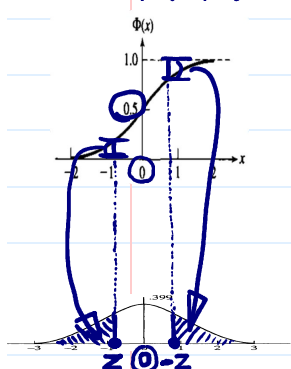
□ Example. Suppose that $X \sim N(\mu, \sigma^2)$. For $-\infty < a < b < \infty$,

$$\begin{aligned} P(a < X < b) &= P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right) \\ &= P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right) \quad \downarrow \\ &= P\left(Z < \frac{b-\mu}{\sigma}\right) - P\left(Z < \frac{a-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \end{aligned}$$

□ Table 5.1 in textbook gives values of Φ .

To read the table:

$\Phi(x)$: cdf of $N(0,1)$



$\phi(x)$: pdf of $N(0,1)$

$$(\Phi(z) + \Phi(-z))/2 = 0.5$$

$$\begin{aligned} \Phi(0) &= \frac{1}{2} \\ \Phi(0.22) &= 0.5871 \\ \Phi(3.36) &= 0.9996 \end{aligned}$$

1. Find the first value of x up to the first place of decimal in the left hand column.

2. Find the second place of decimal across the top row.

3. The value of $\Phi(x)$ is where the row from the first step and the column from the second step intersect.

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

◆ For the values greater than $z=3.49$, $\Phi(z) \approx 1$.

◆ For negative values of z , use $\Phi(z) = 1 - \Phi(-z)$

gamma $\leftrightarrow$ exponential

$Z = X_1 + X_2 + \dots + X_n$, n is large (e.g., binomial $\leftrightarrow$ Bernoulli, negative binomial $\leftrightarrow$ geometric,

Normal distribution plays a central role in the limit theorems of probability (e.g., Central Limit Theorem, CLT, chapter 8)

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$$(L_{Np.5-43 \sim 44}, Ni \uparrow \infty, Ri \uparrow \infty, \frac{Ri}{Ni} \rightarrow p)$$

sum of n
independent Bernoulli(p)

$$X_n \sim B(50, 0.7)$$
$$X_n = 0, 1, 2, 3, \dots$$
$$X'_n = 0, 0.5, 1, 1.5, \dots$$

Q: $X'_n = X_n/2$, $Y'_n = Y_n/2$.
 pmf $\approx$ pdf? (No)
 cdf $\approx$ cdf? (Yes)
 $P(a < X'_n \leq b) = ?$

$$F_{Z_n}(z) = P(Z_n \leq z) \rightarrow \Phi(z).$$

The figure consists of four subplots arranged in a 2x2 grid, each showing a histogram of a binomial distribution and a normal distribution curve. The plots are labeled with their respective parameters:

- Top Left:** Binomial distribution $B(5, 0.7)$ with $\mu_n = 3.5$ and $\sigma_n^2 = 1.05$. The normal distribution is $N(3.5, 1.05)$. The shaded area is between $x=3$ and $x=4$.
- Top Right:** Binomial distribution $B(10, 0.7)$ with $\mu_n = 7$ and $\sigma_n^2 = 2.1$. The normal distribution is $N(7, 2.1)$. The shaded area is between $x=6$ and $x=8$.
- Bottom Left:** Binomial distribution $B(20, 0.7)$ with $\mu_n = 14$ and $\sigma_n^2 = 4.2$. The normal distribution is $N(14, 4.2)$. The shaded area is between $x=12$ and $x=14$.
- Bottom Right:** Binomial distribution $B(50, 0.7)$ with $\mu_n = 35$ and $\sigma_n^2 = 10.5$. The normal distribution is $N(35, 10.5)$. The shaded area is between $x=32$ and $x=38$.

As the number of trials n increases from 5 to 50, the binomial distribution becomes more symmetric and the normal approximation becomes more accurate.

□ $Z_n \stackrel{d}{\approx} \underbrace{(Y_n - \mu_n) / \sigma_n}_{\sim N(0,1)} \xleftrightarrow{\text{c.f.}} Z_n \approx (Y_n - \mu_n) / \sigma_n$

p. 6-37

22. For p close to zero or one $\leftarrow$ skewed
 $\Rightarrow$ require much larger n

Q: Poisson (λ)
 $\stackrel{d}{\approx}$ Normal ?
Ans. Yes, when
 λ is large.
 ss
 np

□ **Q:** Why need continuity correction?

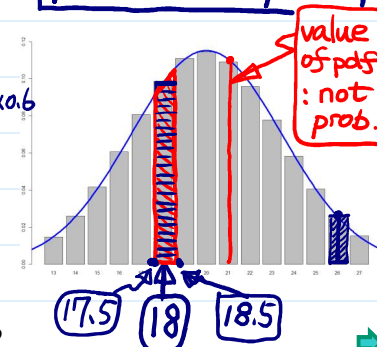
Ans. The binomial(n, p) is a discrete r.v. and we are approximating it with a continuous r.v.

- ◆ For example, suppose $X \sim \text{binomial}(50, 0.4)$ and we want to find $P(X=18)$, which is larger than 0.

- ◆ With the normal r.v. $\underline{Y} \sim N(20, 12)$, however, $P(Y=18)=0$ because $\underline{Y}$ has a continuous distribution

□ Instead, we make a continuity correction,

cf. (height)
pmf: value $\rightarrow$ prob.
pdf: area $\rightarrow$ prob.



How about using the value of pdf $f_Y(18)$?

$$P(X = 18) = P(17.5 < X < 18.5)$$

$$= P\left(\frac{17.5 - (50 \cdot 0.4)}{\sqrt{50 \cdot 0.4 \cdot 0.6}} < Z_n < \frac{18.5 - (50 \cdot 0.4)}{\sqrt{50 \cdot 0.4 \cdot 0.6}}\right)$$

by CLT $\approx P\left(\frac{17.5 - (50 \cdot 0.4)}{\sqrt{50 \cdot 0.4 \cdot 0.6}} < Z < \frac{18.5 - (50 \cdot 0.4)}{\sqrt{50 \cdot 0.4 \cdot 0.6}}\right)$

$\approx P\left(-\frac{2.5}{\sqrt{12}} < Z < -\frac{1.5}{\sqrt{12}}\right) = P\left(Z < -\frac{1.5}{\sqrt{12}}\right) - P\left(Z < -\frac{2.5}{\sqrt{12}}\right)$

$= \Phi\left(-\frac{1.5}{\sqrt{12}}\right) - \Phi\left(-\frac{2.5}{\sqrt{12}}\right) = \left(1 - \Phi\left(\frac{1.5}{\sqrt{12}}\right)\right) - \left(1 - \Phi\left(\frac{2.5}{\sqrt{12}}\right)\right)$

$= \Phi\left(\frac{2.5}{\sqrt{12}}\right) - \Phi\left(\frac{1.5}{\sqrt{12}}\right)$

and can obtain the approximate value from Table 5.1.

□ Similarly,

$$P(X \geq 30) = P(X > 29.5) = P\left(Z_n > \frac{29.5 - (50 \cdot 0.4)}{\sqrt{12}}\right)$$

by CLT $\approx P(Z > 9.5/\sqrt{12}) = 1 - \Phi(9.5/\sqrt{12})$

and $\approx P(Z > 9.5/\sqrt{12}) = 1 - \Phi(9.5/\sqrt{12})$

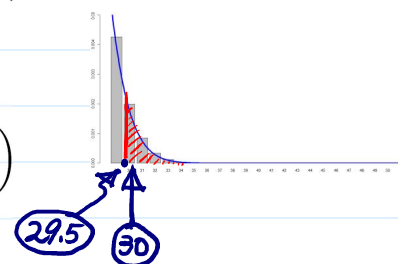
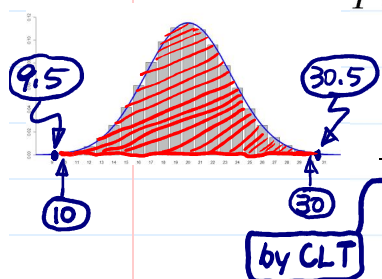
$$P(10 \leq X \leq 30) = P(9.5 < X < 30.5)$$

$$= P\left(\frac{9.5 - (50 \cdot 0.4)}{\sqrt{12}} < Z_n < \frac{30.5 - (50 \cdot 0.4)}{\sqrt{12}}\right)$$

$$\approx P\left(-10.5/\sqrt{12} < Z < 10.5/\sqrt{12}\right)$$

$$= \Phi(10.5/\sqrt{12}) - \Phi(-10.5/\sqrt{12})$$

$$= 2 \cdot \Phi(10.5/\sqrt{12}) - 1 = 1 - \Phi\left(\frac{10.5}{\sqrt{12}}\right)$$



➤ Summary for $X \sim \text{Normal}(\mu, \sigma^2)$

p. 6-39

- Pdf: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$
- Cdf: no close form, but usually denoted by $\Phi((x-\mu)/\sigma)$.
- Parameters: $\mu \in \mathbb{R}$ and $\sigma > 0$.
- Mean: $E(X) = \mu$.
- Variance: $\text{Var}(X) = \sigma^2$.

• Weibull Distribution

For $\alpha, \beta > 0$ and $v \in \mathbb{R}$, the function

fixed constants $f(x) = \begin{cases} \frac{\beta}{\alpha} \left(\frac{x-v}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-v}{\alpha}\right)^\beta}, & \text{if } x \geq v, \\ 0, & \text{if } x < v, \end{cases}$

is a pdf since (1) $f(x) \geq 0$ for all $x \in \mathbb{R}$, and (2)

$$\int_{-\infty}^{\infty} f(x) dx = \int_v^{\infty} \frac{\beta}{\alpha} \left(\frac{x-v}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-v}{\alpha}\right)^\beta} dx$$

$$\stackrel{\text{pdf of exponential(1)}}{=} \int_0^{\infty} e^{-y} dy = -e^{-y} \Big|_0^{\infty} = 1.$$

- The distribution of a random variable X with this pdf is called the Weibull distribution with parameters α, β , and v .

$$y = \left(\frac{x-v}{\alpha}\right)^\beta \Rightarrow x = \alpha y^{\frac{1}{\beta}} + v$$

$$\frac{dx}{dy} = \frac{\alpha}{\beta} y^{\frac{1}{\beta}-1} \Rightarrow dx = \frac{\alpha}{\beta} y^{\frac{1}{\beta}-1} dy$$

possible values of x

(Δ)

➤ (exercise) The cdf of Weibull distribution is

by (Δ) in LNp.6-39

$$F(x) = \begin{cases} 1 - e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta}, & \text{if } x \geq \nu, \\ 0, & \text{if } x < \nu. \end{cases}$$

➤ Theorem. The mean and variance of a Weibull distribution with parameters α , β , and ν are

$$\mu = \alpha \Gamma\left(1 + \frac{1}{\beta}\right) + \nu \quad \text{and}$$

$$\sigma^2 = \alpha^2 \left\{ \Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \right\}.$$

Proof. $E(X) = \int_{\nu}^{\infty} x \cdot \frac{\beta}{\alpha} \left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta} dx$

$$\stackrel{\downarrow}{=} \int_0^{\infty} (\alpha y^{1/\beta} + \nu) e^{-y} dy$$

$$= \alpha \int_0^{\infty} y^{(1/\beta + 1) - 1} e^{-y} dy + \nu \int_0^{\infty} e^{-y} dy = \alpha \Gamma\left(\frac{1}{\beta} + 1\right) + \nu$$

$$E(X^2) = \int_{\nu}^{\infty} x^2 \cdot \frac{\beta}{\alpha} \left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^\beta} dx \quad \text{pdf of exponential(1)}$$

$$\stackrel{\downarrow}{=} \int_0^{\infty} (\alpha y^{1/\beta} + \nu)^2 e^{-y} dy$$

$$= \alpha^2 \int_0^{\infty} y^{(2/\beta + 1) - 1} e^{-y} dy + 2\alpha\nu \int_0^{\infty} y^{(1/\beta + 1) - 1} e^{-y} dy + \nu^2 \int_0^{\infty} e^{-y} dy$$

$$= \alpha^2 \Gamma\left(\frac{2}{\beta} + 1\right) + 2\alpha\nu \Gamma\left(\frac{1}{\beta} + 1\right) + \nu^2$$

➤ Some properties

■ Weibull distribution is widely used to model lifetime (cf., exponential)

memoryless

■ α : scale parameter; β : shape parameter;

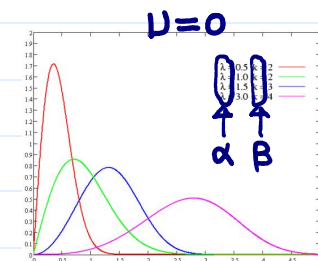
ν : location parameter

■ Theorem. If $X \sim \text{exponential}(\lambda)$, then

$$Y = \alpha (\lambda X)^{1/\beta} + \nu$$

Note: $\lambda X \sim \text{exponential}(1)$

is distributed as Weibull with parameters α , β , and ν (exercise).



Hint: $X \sim \text{Gamma}(\alpha, \lambda) \Rightarrow aX \sim \text{Gamma}(\alpha, \frac{\lambda}{a}), a > 0$ (LNp.6-26)

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• Cauchy Distribution

➤ For $\mu \in \mathbb{R}$ and $\sigma > 0$, the function

$$f(x) = \frac{\sigma}{\pi} \cdot \frac{1}{\sigma^2 + (x-\mu)^2}, \quad -\infty < x < \infty,$$

fixed constants

(Δ)

$$y = \frac{x-\mu}{\sigma}$$

$$\Rightarrow x = \sigma y + \mu$$

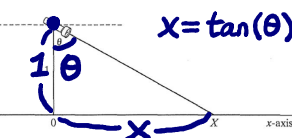
$$\frac{dx}{dy} = \sigma$$

$$\Rightarrow dx = \sigma dy$$

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{\sigma}{\pi} \frac{1}{\sigma^2 + (x-\mu)^2} dx$$

$$\stackrel{\text{pdf of Cauchy}(0,1)}{=} \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+y^2} dy = \frac{1}{\pi} \tan^{-1}(y) \Big|_{-\infty}^{\infty} = 1.$$

★



$\theta \sim \text{Uniform}(-\frac{\pi}{2}, \frac{\pi}{2})$

Let $X = \tan(\theta)$, then

$X \sim \text{Cauchy}(0,1)$

(exercise)

- not mean $\xrightarrow{\quad}$ $\xleftarrow{\quad}$ not standard deviation

Recall. The 2
Thms in LNP.6-8
& 6-9, and ★
in LNP.6-41.

$$F(x) = \int_{-\infty}^x \frac{\sigma}{\pi} \frac{1}{\sigma^2 + (y-\mu)^2} dy = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left(\frac{x-\mu}{\sigma} \right)$$

for $-\infty < x < \infty$. (exercise) by (∇) in LNP. 6-41.

- The mean and variance of Cauchy distribution do not exist because the integral does *not converge absolutely*

► Some properties

Why? check the graph ★
in LNp.6-41

- Cauchy is a heavy tail distribution

- μ : location parameter; σ : scale parameter

■ Theorem. If $X \sim \text{Cauchy}(\mu, \sigma)$, then

$$aX+b \sim \text{Cauchy}(a\mu+b, |a|\sigma).$$

Proof. (exercise)

-Thm (LN p.6-10)

Note: a pdf $f(x) \rightarrow 0$, when $x \rightarrow \infty$ or $x \rightarrow -\infty$.

But, $f(x) \downarrow 0$ how fast?

❖ **Reading:** textbook, Sec 5.4, 5.5, 5.6

