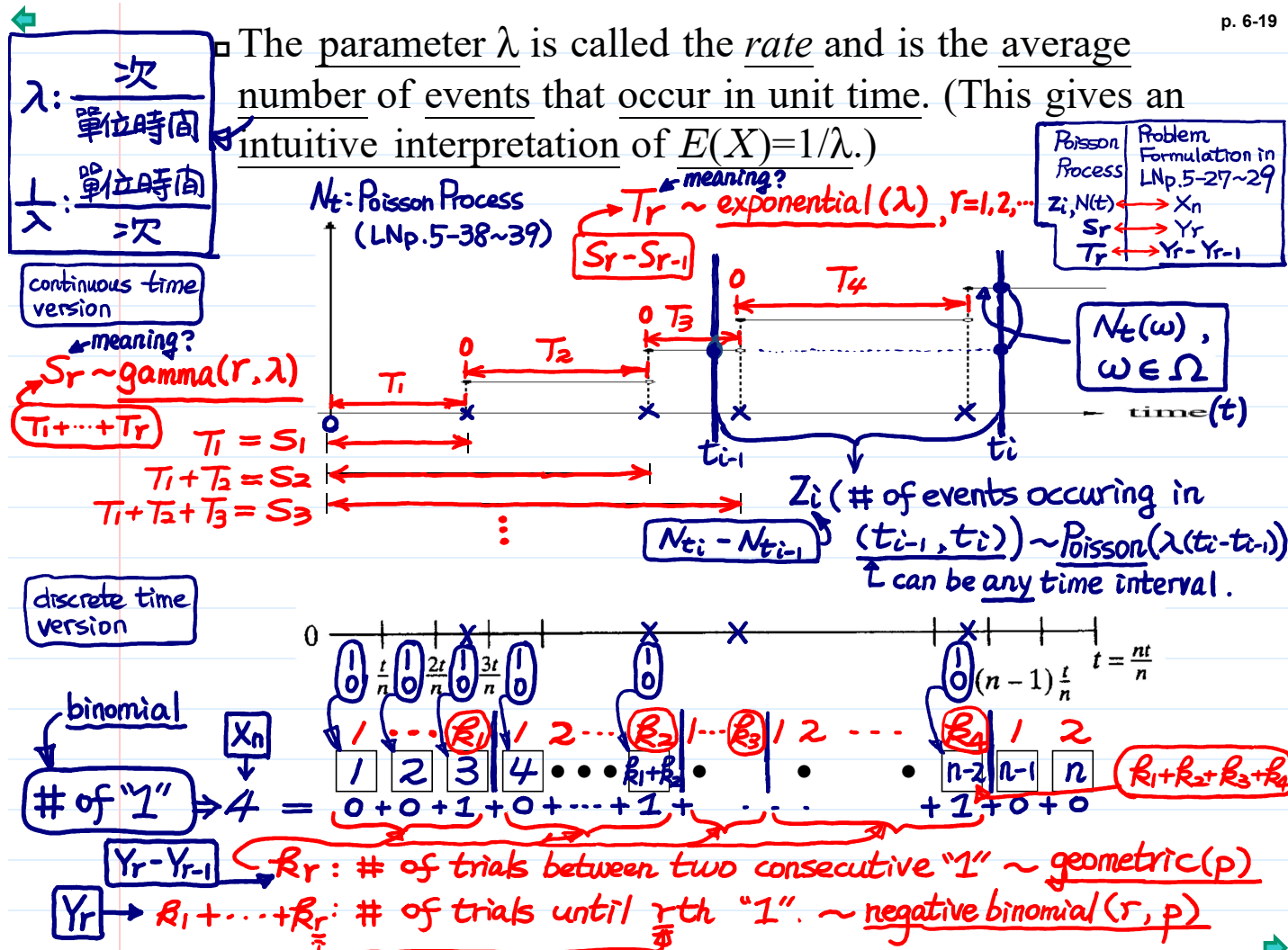
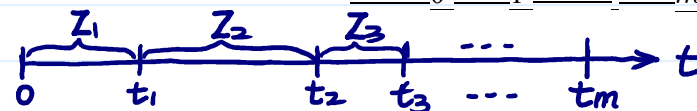




Theorem (relationship between exponential, gamma, and Poisson distributions, Sec. 9.1, textbook). Let

- Problem Formulation in LNp.5-27~28 (C.f.)
1. $T_1, T_2, T_3, \dots$ be independent and $\sim \text{exponential}(\lambda)$,
 2. $S_r = T_1 + \dots + T_r, r=1, 2, 3, \dots$,
 3. Z_i be the number of S_r 's that falls in the time interval $(t_{i-1}, t_i], i=1, \dots, m$, where $0 = t_0 < t_1 < \dots < t_m < \infty$.

Then,



(i) $S_r \sim \text{gamma}(r, \lambda)$. $\leftarrow$ prove in LNp.7-33

(ii) $Z_1, \dots, Z_m$ are independent,

(iii) $Z_i \sim \text{Poisson}(\lambda(t_i - t_{i-1}))$,

(iv) The reverse statement is also true.

1. $Z_1, \dots, Z_m, \dots$: independent & $Z_i \sim \text{Poisson}(\lambda(t_i - t_{i-1}))$
2. define $S_r \rightarrow$ define T_r
3. Then, T_r indep. exponential(λ)

The rate parameter λ is the same for the Poisson, exponential, and gamma random variables.

The exponential distribution can be thought of as the continuous analogue of the geometric distribution.

$$\begin{aligned}
 N_t &= Z \text{ on } [0, t] \\
 &\sim \text{Poisson}(\lambda t) \\
 1 - F_T(t) &= P(T_1 > t) = P(Z=0) \\
 &= \frac{e^{-\lambda t} (\lambda t)^0}{0!} \\
 &= e^{-\lambda t}
 \end{aligned}$$

Theorem. The exponential distribution (like the geometric distribution) is memoryless, i.e., for $s, t \geq 0$,

$$P(X > s+t | X > s) = P(X > t).$$

where $X \sim \text{exponential}(\lambda)$.

exponential(λ) $X_2 \sim \text{exponential}(\lambda)$ p. 6-21

The reverse is also true.

e.g. $X = \text{lifetime} \sim \text{exponential}$

Compare the difference: ① $X_1 = X_2$ ② $X_1 \text{ dist.} = X_2 \text{ dist.}$ ($X_1 \sim X_2$)

Proof.

$$P(X > s+t | X > s) = \frac{P(\{X > s+t\} \cap \{X > s\})}{P(\{X > s\})} = \frac{P(\{X > s+t\})}{P(\{X > s\})}$$

$$= \frac{1 - F_X(s+t)}{1 - F_X(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

by the cdf given in LNp.6-18

Intuitive explanation: check the graph in LNp.6-19.

- This means that the distribution of the waiting time to the next event remains the same regardless of how long we have already been waiting.

Q: Why gamma negative binomial with $r > 1$ does not possess the memoryless property?

- This only happens when events occur (or not) totally at random, i.e., independent of past history. *connection between exponential & geometric*

- Notice that it does not mean the two events $\{X > s+t\}$ and $\{X > s\}$ are independent.

Summary for $X \sim \text{Exponential}(\lambda)$

- Pdf: $f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$
- Cdf: $F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$
- Parameters: $\lambda > 0$.
- Mean: $E(X) = 1/\lambda$.
- Variance: $\text{Var}(X) = 1/\lambda^2$.

If $\{X > s+t\}$ & $\{X > s\}$ are independent, $P(X > s+t | X > s) \neq P(X > s+t)$

alternative exponential ($\lambda' = 1/\lambda$)

- pdf: $f(x) = \frac{1}{\lambda'} e^{-\frac{x}{\lambda'}}$, if $x \geq 0$.
- cdf: $F(x) = 1 - e^{-\frac{x}{\lambda'}}$, if $x \geq 0$.
- $E(X) = \lambda'$
- $\text{Var}(X) = \lambda'^2$

$\lambda = \frac{1}{\lambda'}$

λ : $\frac{1}{\text{單位時間}}$
 λ' : $\frac{\text{單位時間}}{1}$

Gamma Distribution *Recall. LNp.6-19*

伽瑪

Gamma Function

- Definition. For $\alpha > 0$, the gamma function is defined as

- a recursive formula
- know $\Gamma(\alpha)$ for $\alpha \in (0, 1]$ $\Rightarrow$ know $\Gamma(\alpha)$ on any α

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$ (exercise)

▪ $\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$

check textbook, CH5. Theoretical exercise, 5.21

Proof. By integration by parts,

$$\begin{aligned}\Gamma(\alpha + 1) &= \int_0^\infty \underbrace{x^\alpha}_{f} \underbrace{e^{-x}}_{g'} dx \\ &= -x^\alpha e^{-x} \Big|_0^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x} dx = \alpha \Gamma(\alpha).\end{aligned}$$

■ $\Gamma(\alpha) = (\alpha-1)!$ if α is an integer

Proof. $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) = (\alpha-1)(\alpha-2)\Gamma(\alpha-2) = \dots$
 $= (\alpha-1)(\alpha-2)\dots\Gamma(1) = (\alpha-1)!$

■ $\Gamma(\alpha/2) = \frac{\sqrt{\pi}(\alpha-1)!}{2^{\alpha-1}(\frac{\alpha-1}{2})!}$ if α is an odd integer

Proof. $\Gamma(\frac{\alpha}{2}) = (\frac{\alpha-2}{2})\Gamma(\frac{\alpha}{2}-1) = \dots = \left(\frac{\alpha-2}{2}\right)\left(\frac{\alpha-4}{2}\right)\dots\frac{1}{2}\Gamma(\frac{1}{2})$

■ Gamma function is a generalization of the factorial functions

➤ For $\alpha, \lambda > 0$, the function

fixed $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$

from Gamma function *cf. waiting time (LNp.6-19)* *possible values of X*

is a pdf since (1) $f(x) \geq 0$ for all $x \in \mathbb{R}$, and (2)

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} y^{\alpha-1} e^{-y} dy = 1.$$

$y = \lambda x \Rightarrow x = \frac{1}{\lambda} y$
 $\frac{dx}{dy} = \frac{1}{\lambda} \Rightarrow dx = \frac{1}{\lambda} dy$

■ The distribution of a random variable X with this pdf is called the gamma distribution with parameters α and λ .

p. 6-24
not necessary integers

➤ The cdf of gamma distribution can be expressed in terms of the incomplete gamma function, i.e., $F(x)=0$ for $x < 0$, and for $x \geq 0$,

$$F(x) = \int_0^x \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} dy = \frac{1}{\Gamma(\alpha)} \int_0^{\lambda x} \underbrace{z^{\alpha-1}}_f \underbrace{e^{-z}}_{g'} dz = \frac{1}{\Gamma(\alpha)} \gamma(\alpha, \lambda x)$$

If α is an integer

$F(x) = 1 - \sum_{k=0}^{\alpha-1} \frac{e^{-\lambda x} (\lambda x)^k}{k!}$ (exercise)

➤ Theorem. The mean and variance of a gamma distribution with parameter α and λ are

Intuitive explanation

$\mu = \alpha/\lambda$ and $\sigma^2 = \alpha/\lambda^2$

Proof. $\alpha \cdot (\frac{1}{\lambda})$

$$E(X) = \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} = \frac{\alpha}{\lambda}$$

$$E(X^2) = \int_0^\infty x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\lambda^{\alpha+2}} = \frac{\alpha(\alpha+1)}{\lambda^2}$$

$\text{Var}(T_1 + \dots + T_k) = \sum_{i=1}^k \text{Var}(T_i)$
 $\uparrow$
 $\text{indep. } T_1, \dots, T_k$ (LNp.8-13)

$P(X \leq x) = F_X(x) = 1 - F_Y(\alpha-1)$

$\uparrow$ Gamma $P(Y \geq \alpha)$ $\uparrow$ Poisson (λx)

discrete time version (LNp.6-19)

$P(X \leq n) = P(Y \geq r)$

$\uparrow$ negative binomial (r, p) $\uparrow$ binomial (n, p)

$P(X > n) = P(Y < r)$ (LNp.5-28)

$\uparrow$ pdf of Gamma $(\alpha+2, \lambda)$

■ (exercise) $E(X^k) = \frac{\Gamma(\alpha+k)}{\lambda^k \Gamma(\alpha)}$, for $0 < k$, and

$$E\left(\frac{1}{X^k}\right) = \frac{\lambda^k \Gamma(\alpha-k)}{\Gamma(\alpha)}, \text{ for } 0 < k < \alpha.$$

➤ Some properties

check the graph in LNp.6-19

■ The gamma distribution can be used to model the waiting time until a number of random events occurs

■ When $\alpha=1$, it is exponential(λ) the number = α (integer)

prove in LNp.7-33 or using mgf (chapter 7)

■ $T_1, \dots, T_n$: n independent exponential(λ) r.v.'s

$$\Rightarrow T_1 + \dots + T_n \sim \text{Gamma}(n, \lambda)$$

■ Gamma distribution can be thought of as a continuous analogue of the negative binomial distribution

■ A summary (check LNp.6-19)

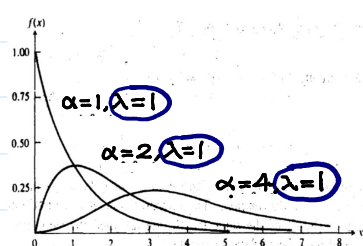
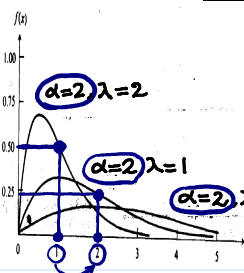
memoryless

	Discrete Time Version	Continuous Time Version
number of events	binomial	Poisson
waiting time until 1st event occurs	geometric	exponential
waiting time until r th events occur	negative binomial	gamma

■ α is called shape parameter and λ scale parameter (Q: how to interpret α and λ in terms of waiting time?)

p. 6-26

$X \sim \text{Gamma}(\alpha, \lambda)$
 $aX \sim \text{Gamma}(\alpha, \frac{\lambda}{a})$
 for $a > 0$.
 (exercise, can use the Thm in LNp.6-10)



α : X is waiting time until α th occurrence
 λ : 次/單位時間

FYI

■ A special case of the gamma distribution occurs when $\alpha=n/2$ and $\lambda=1/2$ for some positive integer n . This is known as the Chi-squared distribution with n degrees of freedom (Chapter 6)

eg. $X_1 \rightarrow \lambda_1$: 次/天
 $X_2 \rightarrow \lambda_2$: 次/11時
 $X_3 \rightarrow \lambda_3$: 次/分鐘
 $\lambda_1 = 24\lambda_2 = 1440\lambda_3$
 $X_1 = \frac{X_2}{24} = \frac{X_3}{1440}$

➤ Summary for $X \sim \text{Gamma}(\alpha, \lambda)$

■ Pdf: $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$

■ Cdf: $F(x) = \gamma(\alpha, \lambda x) / \Gamma(\alpha)$.

■ Parameters: $\alpha, \lambda > 0$.

■ Mean: $E(X) = \alpha / \lambda$.

■ Variance: $\text{Var}(X) = \alpha / \lambda^2$.

definition:

$X_1^2 + \dots + X_n^2$
 where $X_1, \dots, X_n$
 are independent
 and $\sim \text{Normal}(0,1)$